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A Control-theoretic View on Incentives*

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A unifying framework exists for several results about Stackelberg games including incentive problems with special strategy spaces and a relationship with social choice theory.

Key Words—Social and behavioral sciences; team theory; stochastic control; multivariable control systems; game theory; cybernetics; hierarchical systems; incentives.

Abstract—The idea of declaring a reward (punishment) for a decision maker according to his particular choice of action in order to induce certain 'desired' behavior on the part of the decision maker is known as incentive (threat). This practice is age old. However, only in recent years have the notions been formalized. In the development of a control-theoretic view on incentives, we first investigate the deterministic version of the incentive problem. This reveals the basic simple idea behind the problem. It also illustrates the different possibilities introduced by the presence of dynamics and multi-follower nature of the problem. This is followed by two variants of the stochastic version of the problem where we concentrate on the role of uncertainties. Relationship to economic literature is also discussed.

1. INTRODUCTION

THE IDEA of declaring a reward (punishment) for a decision maker according to his particular choice of action in order to induce certain 'desired' behavior on the part of the decision maker is known as incentive (threat). This practice is age old. However, only in recent years have the notions been formalized. Before going into details, let us pursue a bit further the above intuitive motivations.

The following are the minimal ingredients for the incentive problem.

(1) There are at least two decision makers; the leader who declares the incentives, the follower who chooses his acts based on the declared incentives.

(2) The leader's decision is denoted as u_0; the follower's, u_1; and the incentive, γ_0: $u_1 \to u_0$.

(3) From the leader's viewpoint, there is a desired act, u_1^d for the follower. It is possible but not necessary that $u_1 = u_1^d$ minimizes some $L_0(u_1)$ which is the payoff for the leader.

(4) The follower is an optimizer who is interested in minimizing some $L_1(u_0, u_1)$ knowing γ_0. Note that dependence of L_1 on u_0 is necessary for the incentive problem to be meaningful.

In (1)–(4) we have been deliberately vague concerning the specifications of the domains and ranges for L_0, L_1, u_0 and u_1; the information z_0, z_1 available to the leader and the follower respectively when they choose their decisions; and the presence of uncertainties and dynamics.

Roughly speaking, the literature on the subject can be divided into two groups. From the control theory side, emphasis has been almost exclusively on deterministic problems involving dynamics, see for instance Başar and Selbuz (1979a); Tolwinski (1981); Papavassilipoulos and Cruz (1979, 1980). Some stochastic problems have been treated in Başar (1980b). In other words, u_0 and u_1 in the above problem represent controls applied to dynamical systems in time. Subject to causality constraints, a variety of dependences of u_0 on u_1 can be studied. More will be said in Section 2. On the mathematical economic front, a great deal of literature exists concerning uncertainties in incentive problems (Groves and Loeb, 1979; Dasgupta, Hammond and Maskin, 1979). Briefly, the follower is supposed to possess information z_1 about the state of nature, ξ, which is unknown but useful to the leader. The leader must induce the follower to reveal z_1 via u_1 by proper choice of the incentives $u_0 = \gamma_0(u_1) = \gamma_0[\gamma_1(z_1)]$ where γ_1 is the strategy chosen by the follower. Allocation of public good is a prime example of such problems.[1]

In Section 2 we shall first study the deterministic version of the incentive problem. This will reveal the basic simple idea behind the

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[1] In this case, L_0 is interpreted as the social welfare criterion; L_1, the individual utility; z_1, the follower's preference; u_1, the reported preference of the follower which need not be the true z_1.

problem. It will also illustrate the different possibilities introduced by the presence of dynamics and multi-follower nature of the problem. This will be followed by two variants of the stochastic version of the problem in Sections 3 and 4 where we shall concentrate on the role of uncertainties. Relationship to economic literature will be discussed in Section 5.

2. DETERMINISTIC INCENTIVE PROBLEMS

2.1. *The basic approach*

Let u_0 and u_1 take values in $U_0 \subset R^{n_0}$ and $U_1 \subset R^{n_1}$, respectively; L_0: $R^{n_0} \cdot R^{n_1} \to R$; L_1: $R^{n_0} \cdot R^{n_1} \to R$. We define the desired choice for the leader as

$$(u_0^d, u_1^d) = \arg\min_{u_0, u_1} L_0(u_0, u_1). \quad (1)$$

The incentive problem can then be simply stated as:*

(P-1). Find γ_0: $U_1 \to U_0$, $\gamma_0 \in \Gamma_0$ such that

$$\arg\min_{u_1} L_1[\gamma_0(u_1), u_1] = u_1^d \quad (2a)$$

$$\gamma_0(u_1^d) = u_0^d \quad (2b)$$

where Γ_0 is the class of admissible incentives.

Note that (2a) and (2b) require choosing a set of n_0 functions γ_0: $U^1 \to U^0$ to satisfy $n_0 + n_1$ equations. If the set of n_0 functions has $n_0 + n_1$ or more undetermined parameters then we might in general accomplish this by choosing the parameters appropriately. However, before proceeding on this task, let us dispose of one naive approach, i.e. using an 'infinite threat'. Take the simple example of $L_0 = u_0^2 + u_1^2$, $L_1 = (u_0 - 1)^2 + (u_1 - 1)^2$ where u_0 and u_1 are scalars. By inspection, $u_1^d = u_0^d = 0$. Consider $u_0 = ku_1$ with k approaching infinity as a possible incentive mechanism. The idea is that any choice of $u_1 \neq 0$ will make L_1 approach infinity and thus force u_1 to approach u_1^d in his own interest. However by substituting $u_0 = ku_1$ into L_1, it is easily shown that $u_1 = (k+1)/(k^2+1)$ and $u_0 = ku_1 = (k^2+k)/(k^2+1)$. u_1 approaches 0 ($= u_1^d$), u_0 approaches 1 ($\neq u_0^d$) as k approaches infinity, and (2b) is violated. Consequently 'infinite threat' as described above is not generally feasible. More elaborate examples can be constructed to show that infinite threat will not always work. Furthermore, such a threat may not be credible in practice. Similarly incentives such as

$$u_0 = u_0^d \text{ for } u_1 = u_1^d$$
$$u_0 = \text{infinite for } u_1 \neq u_1^d$$

can be ruled out if we suitably restrict the class of admissible incentives, e.g. Γ_0 must contain only continuous maps.

Returning to the problem of choosing γ_0 to satisfy (2), let us consider an incentive γ_0 of the form

$$u_0 = u_0^d + g(u_1, u_1^d) \quad (3)$$

where $g(u_1^d, u_1^d) \equiv 0$. In particular, let $g = k(u_1 - u_1^d)$. Formula (3) automatically satisfies (2b), thus we only need to choose g or k to satisfy (2a). For this example equation (2a) reduces to $(k+1)/(k^2+1) = 0$ or, equivalently, $k = -1$. With this choice of k, we note that $L_1(ku_1, u_1) = 2u_1^2 + 2 = L_0(u_0^d, u_1) + 2$. In other words, by this choice of incentive, we have simply made the objectives of the follower and leader essentially the same thus fulfilling the old adage

> "If you wish other people to behave in your interest, then make them see things your way". (4)

This self-evident truth is the heart of the incentive mechanism. We shall come back to it many times. Following (2a) we shall in fact generalize (2a) as

$$\arg\min_{u_1} L_1[\gamma_0(u_1), u_1] = \arg\min_{u_1} L_0(u_0^d, u_1). \quad (2a')$$

Note that (2a') does *not* necessarily require $L_1[\gamma_0(u_1), u_1]$ and $L_0(u_0^d, u_1)$ to be identical within linear transformation as in the above example. It should also be clear from the above discussion that there is considerable (almost unlimited) freedom in the choice of γ_0 that will satisfy (2a'). This non-uniqueness can also be visualized graphically. Let us return to the example mentioned above. In Fig. 1(a) the L_1 contours have been drawn. The desired point (u_0^d, u_1^d) is the origin. By announcing $u_0 = \gamma_0(u_1) = -u_1$, the leader ensures that the solution (u_0, u_1) will lie on the line $u_0 = -u_1$ in the (u_0, u_1) space independent of the action of the follower. Being rational, the follower will choose the point on this line which minimizes his cost function; such a choice is $u_1 = 0$. We shall say that the problem is *incentive controllable* (i.c.) since (u_0^d, u_1^d) can be realized.

Note that the line $u_0 = -u_1$ is not the only

*It is assumed that (u_0^d, u_1^d) derived from the minimization process in (1) is unique.

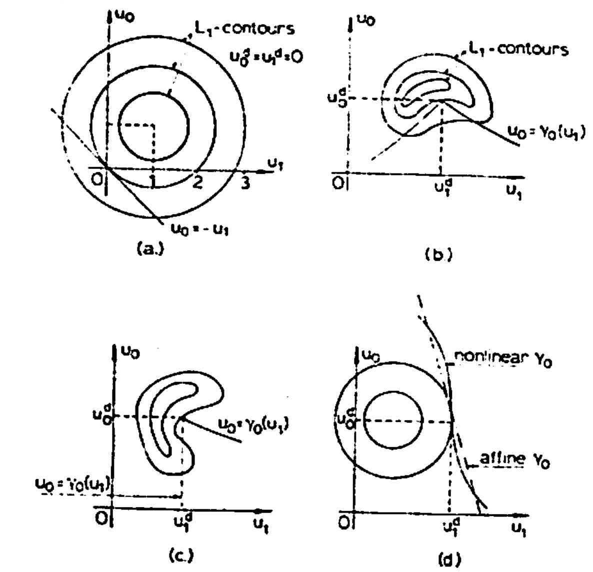

FIG. 1. Graphical display of various forms of incentive controllability.

curve which realizes (u_0^d, u_1^d); any curve $u_0 = \gamma_0(u_1)$ with $0 = \gamma_0(0)$ and with its graph outside the L_1 contour through (0, 0), will do the same. Since in the above problem an affine $u_0 = \gamma_0(u_1)$ exists for which (u_0^d, u_1^d) can be achieved, we call it *linearly incentive controllable* (*l.i.c.*). The problem depicted in Fig. 1(b) is i.c., but not l.i.c.; that is the curve $u_0 = \gamma_0(u_1)$ through (u_0^d, u_1^d), and with its graph outside the L_1 contour through this point, can not be linear. In Fig. 1(c) a problem has been depicted that it is not even *continuously i.c.* (a problem is continuously i.c. if a continuous curve $u_0 = \gamma_0(u_1)$ exists such that (u_0^d, u_1^d) can be achieved). In Fig. 1(d), the problem is continuously i.c. but not l.i.c. The problem is *ϵ-l.i.c.* in the sense that the leader can not force the joint decision to (u_0^d, u_1^d) by an affine γ_0, however, he can get arbitrarily close to it by an affine γ_0. An example of this will be presented in Section 2.2. In Section 5, a problem will be encountered which is not i.c. at all according to its definition given above; it is, however, *mixed i.c.* (by announcing a 'mixed' strategy γ_0, (u_0^d, u_1^d) can be achieved).

2.2. *Extension to multi-stage case*

Consider a T-stage problem with decision variables $u_{0,0}, u_{0,1}, \ldots, u_{0,T-1}; u_{1,0}, u_{1,1}, \ldots, u_{1,T-1}$. The first index indicates decision maker and the second index refers to time. The idea is that at time t, the leader can choose his decision $u_{0,t}$ based on past decisions of the follower, thus incentives can be imputed to all $u_{1,t}$ except the last $u_{1,T-1}$. For the same reason $u_{0,0}$ cannot be used to provide incentives. Thus without loss of generality we shall assume $u_{0,0}$ and $u_{1,T-1}$ to be constants or fixed strategies determined by considerations outside the incentive problem. Now let $\gamma_0^d \equiv (\gamma_{0,1}^d, \ldots, \gamma_{0,T-1}^d)^T$ and $\gamma_1^d \equiv (\gamma_{1,0}^d, \ldots, \gamma_{1,T-2}^d)^T$ be the desired sequences of strategies which are obtained as

$$(\gamma_0^d, \gamma_1^d) = \arg\min_{\gamma_0, \gamma_1} L_0(\gamma_0, \gamma_1). \tag{5}$$

Let $u_{0,t}^d$, $u_{1,t}^d$ be the corresponding desired sequences of decisions; and L_0^d be the associated desired cost.* Because of the requirement of causality, the general form of incentives is

$$\begin{aligned} u_{0,1} &= \gamma_{0,1}(u_{1,0}) \\ u_{0,2} &= \gamma_{0,2}(u_{1,0}, u_{1,1}) \\ &\vdots \\ u_{0,T-1} &= \gamma_{0,T-1}(u_{1,0}, u_{1,1}, \ldots, u_{1,T-2}). \end{aligned} \tag{6}$$

A simple special case of (6) is

$$u_{0,t} = \gamma_{0,t}(u_{1,t-1}) \quad \text{for } t \text{ from 1 to } T-1 \tag{7}$$

which is simply a concatenation of single-stage problems. Another simple example would be to require

$$\begin{aligned} u_{0,i} &= \text{constant} \quad \forall\ i \neq T-1 \\ u_{0,T-1} &= \gamma_{0,T-1}(u_{1,0}, \ldots, u_{1,T-2}) \end{aligned} \tag{8}$$

which has the interpretation that all rewards (punishments) are given at the end based on the entire decision history of the follower.

A more usual situation is to use the concept of 'state' which summarizes past decision histories of the system. Let x_t be the state at time t, $x_t = f_{t-1}(x_{t-1}, u_{0,t-1}, u_{1,t-1})$, then x_t is a function of initial state x_0 and all past decisions. We shall examine two variations of (6).

Variation 1. Suppose we have γ_0^d and γ_1^d as the desired sequences of strategies which generate the desired sequences of decisions u_0^d and u_1^d. In the spirit of (3), let

$$u_{0,t} = \gamma_{0,t}^d(x_t) + k_t \cdot (x_t - \bar{x}_t) \quad \text{for } t \text{ from 1 to } T-1 \tag{9}$$

where $\bar{x}_t \equiv f_{t-1}[x_{t-1}, \gamma_{0,t-1}^d(x_{t-1}), \gamma_{1,t-1}^d(x_{t-1})]$ (i.e. the state at time t if both the decision makers

*In deterministic, single-stage case the distinction between the strategy, γ, and the value taken on by γ during a particular realization, u, is not crucual. In deterministic, multi-stage case, using γ permits us to discuss both open and closed-loop solutions simultaneously. It also prepares the way for Sections 3 and 4.

used the desired strategies at $t-1$).* Note that x_t is a function of $u_{1,j}$, $j<t$ for fixed γ_0^d, thus formula (9) is a special case of formula (6). If we can find k_j for all j such that the follower's optimal strategies, taking into account formula (9), is $\gamma_{1,0}^d, \ldots, \gamma_{1,T-2}^d$, then $u_{0,t} = u_{0,t}^d$ and $u_{1,t} = u_{1,t}^d$ for all t. The desired sequences of decisions can thus be realized.

In Başar and Selbuz (1979a), the authors derived a set of sufficient conditions for the existence of k_t, which are assumed to be constants, for linear system with quadratic criteria. The advantage of formula (9) is that since k_t is a constant, $u_{0,t}$ is affine in its information. The disadvantage is that if $u_{1,t} \neq \gamma_{1,t}^d\ (x_t)$ at any t, then in general we have $x_j - \bar{x}_j \neq 0$ for $j>t$. That is, the follower will be punished forever once a deviation, however unintentional, is observed, even if he returns to γ_1^d afterwards [also see Tolwinski (1981)].

Example 1. Consider the two-person three-stage dynamic game described by the state equations

$$x_1 = x_0 + u_{0,0} + u_{1,0}$$
$$x_2 = x_1 + u_{0,1} + u_{1,1}$$
$$x_3 = x_2 + u_{0,2}$$

and the quadratic cost functions

$$L_0 = x_3^2 + 2u_{0,2}^2 + 2u_{0,1}^2 + 2u_{1,1}^2 + u_{0,0}^2 + u_{1,0}^2$$
$$L_1 = x_3^2 + u_{0,2}^2 + u_{0,1}^2 + bu_{1,1}^2 + 2u_{1,0}^2$$

where b is some constant, $b>0$ and $b \neq 1$. The desired solution for the leader can easily be obtained

$$\gamma_{0,0}^d = -\tfrac{2}{9}x_0, \qquad \gamma_{0,1}^d = -\tfrac{1}{5}x_1, \qquad \gamma_{0,2}^d = -\tfrac{1}{3}x_2,$$
$$\gamma_{1,0}^d = -\tfrac{2}{9}x_0, \qquad \gamma_{1,1}^d = -\tfrac{1}{5}x_1.$$

We now define $\bar{x}_i$ as the state of the system at stage i when γ_0 and γ_1 follow the desired closed-loop strategies above; $\bar{x}_1 = \tfrac{5}{9}x_0$, $\bar{x}_2 = \tfrac{3}{5}x_1$. The desired sequence of decisions can be realized by the following incentives

$$u_{0,1} = -\tfrac{1}{5}x_1 + k_1(x_1 - \bar{x}_1)$$
$$u_{0,2} = -\tfrac{1}{3}x_2 + k_2(x_2 - \bar{x}_2)$$

where $k_1 \equiv (14-b)/5(b-1)$, $k_2 = b - \tfrac{1}{3}$. Note that if $u_{1,0} = -\tfrac{2}{9}x_0 + e$ $(e \neq 0)$, then $u_{0,1}$ and $u_{0,2}$ will not be equal to $-\tfrac{1}{5}x_1$ and $-\tfrac{1}{3}x_2$, respectively, even if $u_{1,1} = -\tfrac{1}{5}x_1$.

Now let us consider the case when $b=1$, for which the above incentive mechanism is not well defined. Some rather straightforward, but extensive manipulations show that for $k_1 \to +\infty$ or $k_1 \to -\infty$, the leader approaches his desired cost L_0^d. We have a situation as depicted in Fig. 1(d); i.e. the problem is only ϵ-l.i.c. In Fig. 1(d) the problem is continuously i.c., and it turns out to be the case here too. If the leader chooses

$$\gamma_{0,1} = -\tfrac{1}{5}x_1 + c|x_0|\sqrt{(|x_1 - \bar{x}_1|)}\ \mathrm{sgn}(x_1 - \bar{x}_1)$$
$$\gamma_{0,2} = -\tfrac{1}{3}x_2 - \tfrac{2}{3}(x_2 - \bar{x}_2)$$

where c is a constant such that $c^2 > 652/675$, he will obtain his desired cost L_0^d.†

Variation 2. Tolwinski (1981) used the following incentive mechanism:

$$u_{0,t} = \gamma_{0,t}^d(x_t) + g_t(x_t - \bar{x}_t) \quad \text{for } t \text{ from 1 to } T-1 \tag{10}$$

where g_t is a nonlinear function with $g_t(0) = 0$ and $\bar{x}_t \equiv f[x_{t-1}, u_{0,t-1}, \gamma_{1,t-1}^d\ (x_{t-1})]$. Note the difference in the definition of $\bar{x}_t$ between variations 1 and 2. In this case, so long as $u_{1,t-1} = \gamma_{1,t-1}^d\ (x_{t-1})$ then $\bar{x}_t = x_t$ regardless whether or not $u_{0,t-1} = \gamma_{0,t-1}^d\ (x_{t-1})$. Thus if the follower acted improperly for whatever reason at $t-2$ but resumes the correct decision at $t-1$ then the leader will only react (punish) at $t-1$ for one step. Thereafter beginning at t, the system is ready to resume the desired sequence of decisions again from whatever the resultant state x_t, and the strategy (γ_0, γ_1^d) is still a solution for the problem considered on the interval $[t, T]$. In fact, it is easy to devise variations of mechanism (10) where the punishment for deviation may last over one, two, three,..., $T-1$ stages.

The above discussion hopefully makes clear that an enormous range of possibilities for incentive exists as special cases of (6). Mechanisms (9) and (10) are the most obvious cases. We should explore others and demand additional properties such as noise immunity, linearity, complexity, uniqueness, etc. to be satisfied with any incentive mechanism. The surface has only been scratched. In fact, it is generally agreed that deterministic formulation of the closed-loop Stackelberg problem is impractical. The slightest bit of noise will destroy the solution. The main reason for expounding this case is to clarify the underlying concept involved which will then be extended to the more realistic stochastic cases.

*By definition of $\bar{x}_t$, the leader needs at least one step memory to implement formula (9).

†This problem is from Basar and Selbuz (1979b), where a two step memory representation was suggested to solve the incentive problem when $b=1$.

2.3. *Extension to many-follower case*

When there are two or more followers in the problem, the relationship among the followers must be specified. We shall illustrate a few of them, assuming coalitions among followers are not allowed. First, let γ_i^d be the desired strategies for decision maker i for i from 0 to m. An incentive mechanism γ_0 is said to induce a *dominant strategy* solution if

$$\arg\min_{\gamma_i} L_i = \gamma_i^d \text{ with arbitrary } \gamma_j, \quad \forall j,\ j \neq i;\quad i = 1,\ldots,m. \tag{11}$$

For example with $u_0 \equiv (u_{01}, u_{02})$, let $L_0 = (u_1+u_2)^2$, $L_1 = (u_1-1)^2 + u_{01}$, $L_2 = (u_2-1)^2 + u_{02}$, then the incentive mechanism $u_{01} = 2u_1$ and $u_{02} = 2u_2$ will induce $u_1 = 0$ regardless of u_2, and similarly for u_2. Dominant strategy solution is the most desirable result since it effectively decouples the followers from each other. However, such solution is very difficult to realize since in general the leader is not that powerful. In the *Nash equilibrium* solution concept, we only require

$$\arg\min_{\gamma_i} L_i = \gamma_i^d \text{ with } \gamma_j = \gamma_j^d, \quad \forall j, j \neq i;\ i = 1,\ldots,m \tag{12}$$

i.e. each agent will behave desirably conditioned on the fact that others will do so. In economics literature, a distinction is made between definition (12) and

$$\arg\min_{\gamma_i} L_i = \gamma_i^d \text{ with } u_j = u_j^d \quad \forall j, j \neq i; i = 1,\ldots,m \tag{12'}$$

where the latter is called the Nash solution while definition (12) is called the *Bayes* solution. This difference becomes significant in the stochastic case. Nash equilibrium [equation (12)] has been studied in Başar and Selbuz (1979a) under dynamic system setup, where the leader's strategy is again of the form (9).

A particular case occurs if the incentive mechanism γ_0 can be chosen such that the cost function of all the followers become identical. Then the followers face a team problem, which has only one 'reasonable' solution. As an example, consider the following example.

Example 2

$$L_0 = u_{01}^2 + u_{02}^2 + u_1^2 + u_2^2$$

$$L_1 = u_{01} - 3u_{02} + (u_1-1)^2 + (u_2-1)^2$$

$$L_2 = u_{01} + u_{02} + (u_1+1)^2 + (u_2+1)^2.$$

The minimum of L_0 is 0 which occurs when all the decisions are zero. If the leader announces

$$u_{01} = \tfrac{3}{4}(u_1-1)^2 + \tfrac{3}{4}(u_2-1)^2 + \tfrac{1}{4}(u_1+1)^2 + \tfrac{1}{4}(u_2+1)^2 - 2$$

$$u_{02} = \tfrac{1}{4}(u_1-1)^2 + \tfrac{1}{4}(u_2-1)^2 - \tfrac{1}{4}(u_1+1)^2 - \tfrac{1}{4}(u_2+1)^2$$

then the followers face the problem

$$\min_{u_1} [(u_1-1)^2 + (u_2-1)^2 + (u_1+1)^2 + (u_2+1)^2 - 2]$$

$$\min_{u_2} [(u_1-1)^2 + (u_2-1)^2 + (u_1+1)^2 + (u_2+1)^2 - 2]$$

which is a team problem, of which the solution is $u_1 = u_2 = 0$. Substituting these values into u_{01} and u_{02} yields $u_{01} = u_{02} = 0$. In this example with two followers, another incentive mechanism exists which makes the followers face a zero-sum game, which also has only one 'reasonable' solution. If the leader announces

$$u_{01} = -\tfrac{3}{4}(u_1-1)^2 + \tfrac{1}{4}(u_2-1)^2 - \tfrac{5}{4}(u_1+1)^2 - \tfrac{1}{4}(u_2+1)^2 + 2$$

$$u_{02} = -\tfrac{1}{4}(u_1-1)^2 + \tfrac{3}{4}(u_2-1)^2 - \tfrac{3}{4}(u_1+1)^2 + \tfrac{1}{4}(u_2+1)^2$$

then the followers face the problem

$$\min_{u_1} [(u_1-1)^2 - (u_2-1)^2 + (u_1+1)^2 - (u_2+1)^2 + 2]$$

$$\min_{u_2} [-(u_1-1)^2 + (u_2-1)^2 - (u_1+1)^2 + (u_2+1)^2 + 2]$$

for which the saddle-point solution is $u_1 = u_2 = 0$. Substituting these values into u_{01} and u_{02} leads to $u_{01} = u_{02} = 0$.

In contrast to the Nash equilibrium concept, there may exist among the followers certain additional levels of hierarchy. For example, DM1 may announce his strategy before the rest of the followers, knowning the leader's strategy. DM1 can thus also implement a kind of incentive mechanism of its own on the rest of the followers. A problem of this sort is said to have *multi-levels of hierarchy*, and has been studied in Başar (1980b) and Tolwinski (1980). Here we shall assume there are three decision makers. DM0 announces his strategy first as a function of the decisions of DM1 and DM2, i.e. u_1 and u_2, respectively. Then DM1 announces his strategy as a function of u_2. A set of sufficient conditions can be stated such that DM1 and DM2 are induced to help DM0 to minimize L_0; also DM2 is induced to help DM1 in minimizing L_1. For any given γ_0 and γ_1, let $\bar{L}_1(u_1, u_2; \gamma_0)$ be derived from $L_1(u_0, u_1, u_2)$ with u_0 being replaced by $\gamma_0(u_1, u_2)$; and $\bar{L}_2(u_2; \gamma_1, \gamma_0)$ be derived from $L_2(u_0, u_1, u_2)$ with u_0 being substituted by $\gamma_0(u_1, u_2)$ and u_1 by $\gamma_1(u_2)$. Define

$$(u_0^{d0}, u_1^{d0}, u_2^{d0}) = \arg\min_{u_0, u_1, u_2} L_0(u_0, u_1, u_2)$$

$$[u_1^{d1}(\gamma_0), u_2^{d1}(\gamma_0)] = \arg\min_{u_1, u_2} \bar{L}_1(u_1, u_2; \gamma_0)$$

$$u_2^{d2}(\gamma_1; \gamma_0) = \arg\min_{u_2} \bar{L}_2(u_2; \gamma_1, \gamma_0).$$

Then a set of sufficient conditions for (γ_0^*, γ_1^*) to achieve the team optimum is

$$u_2^{d2}(\gamma_1^*; \gamma_0^*) = u_2^{d1}(\gamma_0^*) = u_2^{d0}$$
$$u_1^{d1}(\gamma_0^*) = u_1^{d0} = \gamma_1^*(u_2^{d0})$$

and

$$\gamma_0^*(u_1^{d0}, u_2^{d0}) = u_0^{d0}.$$

The desired decision $(u_0^{d0}, u_1^{d0}, u_2^{d0})$ can thus be realized. In Tolwinski (1980) some extra conditions are imposed so that DM1's equilibrium strategy γ_1^* is optimal for him for any choice of $u_2 \in U_2$.

In a three-person game, the relationship among the decision makers can be illustrated graphically. Figure 2(a) shows the one-leader, two-follower model, with the leader being at the top of the structure. The incentive mechanism γ_0 may induce either a dominant strategy solution or a Nash equilibrium solution between the followers. Figure 2(b) shows the case with multi-levels of hierarchy. The conventional Nash solution concept (with no leader, and all the decision makers in symmetric positions) can be depicted by Fig. 2(c). A problem can also be formulated where there are two leaders playing Nash and one follower being dictated by them, as shown by Fig. 2(d).

When coalitions among followers are allowed, the problem becomes more difficult to handle. Even if a dominant strategy exists, the followers may deviate from the desired strategies by forming coalitions. The well-known prisoners' dilemma is an excellent example. We shall not pursue it further here.

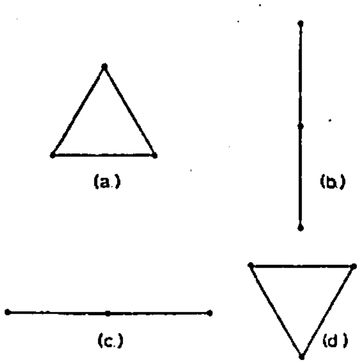

FIG. 2. Relations among decision makers in a three-person game.

3. STOCHASTIC INCENTIVE PROBLEMS I (NESTED CASE)

A natural generalization of the problem in Section 2 is to introduce uncertainties represented by the state of nature, ξ, where $\xi \in \Xi$ and $\Xi \subseteq R^p$. Let the distribution of ξ, $p(\xi)$, be known to the decision makers. We introduce

$$J_0 = E[L_0(u_0, u_1, \xi)] \tag{13}$$

$$J_1 = E[L_1(u_0, u_1, \xi)] \tag{14}$$

as the criteria for the leader and the follower, respectively. The follower is given certain information (measurement, observation) $z_1 \in Z_1 \subseteq R^{m_1}$, where

$$z_1 = \eta_1(\xi). \tag{15}$$

The leader in addition to u_1 also has available information $z_0 \in Z_0 \subseteq R^{m_0}$. Thus his information structure η_0 consists of u_1 and $z_0 \equiv \eta_0(\xi)$. For this section we shall assume

(A1). $\quad \eta_1 \subseteq \eta_0$.

i.e. the follower's information is nested in that of the leader.

Given definition (13), the desired solution is now given not by $(u_0{}^d, u_1{}^d)$ but by strategies

$$[\gamma_0{}^d(z_0, u_1), \ \gamma_1{}^d(z_1)] = \arg\min_{\gamma_0, \gamma_1} E[L_0(u_0 = \gamma_0(z_0, u_1), \quad u_1 = \gamma_1(z_1), \xi)]. \tag{16}$$

Equation (16) denotes a decentralized statistical decision or team problem. Because of (A1), it is reasonable to suppose that $\gamma_0{}^d$, $\gamma_1{}^d$ can be determined (Ho and Chu, 1972). The stochastic version of (P-1) can be stated as

(P-2). Find γ_0: $R^{m_0} \cdot U^1 \to U^0$ with $\gamma_0 \in \Gamma_0$ such that

$$\arg\min_{\gamma_1} E/z_1[L_1(\gamma_0(z_0, \gamma_1), \gamma_1, \xi)] \equiv \gamma_1^d(z_1) \quad (17a)$$

$$\gamma_0[z_0, \gamma_1^d(z_1)] \equiv \gamma_0^d[z_0, \gamma_1^d(z_1)]. \quad (17b)$$

Note that we require (17a) and (17b) to be satisfied for all z_0 and z_1, i.e. they are identities. Since $\gamma_0(z_0, u_1)$ is a function of two variables, it is not unreasonable to expect that it can be chosen to satisfy the two identities (17a) and (17b). In the case when z_0 and z_1 take on discrete values, identities (17a) and (17b) are equivalent to a system of (2a) and (2b) each indexed by a particular pair of (z_0, z_1) values. The analogs of formulas (3) and (2a′) are

$$u_0 = \gamma_0^d(z_0, u_1) + g[z_0, u_1, \gamma_1^d(z_1)] \quad (18)$$

where $g(z_0, u_1^d, u_1^d) \equiv 0$, and

$$\arg\min_{\gamma_1} E[L_1(\gamma_0(z_0, \gamma_1), \gamma_1, \xi)]$$

$$\equiv \arg\min_{\gamma_1} E[L_0(\gamma_0^d(z_0, \gamma_1), \gamma_1, \xi)]. \quad (17a')$$

If we convert (17a′) to its equivalent extensive form then we have

$$\arg\min_{u_1} E/z_1[L_1(\gamma_0(z_0, u_1), u_1, \xi)]$$

$$\equiv \arg\min_{u_1} E/z_1[L_0(\gamma_0^d(z_0, u_1), u_1, \xi)] \quad (17a'')$$

which is equivalent to require

$$\arg\min_{u_1} h_1(z_1, u_1) \equiv$$

$$\arg\min_{u_1} h_0(z_1, u_1) \quad \forall\ z_1 \quad (19)$$

where $h_i \equiv E/z_1[L_i]$, $i = 0, 1$, is given by the obvious definition. Equation (19) says that we require the minimizing function $u_1 = \gamma_1(z_1)$ to be identical for both h_0 and h_1. This requirement can be given a different characterization which is useful in certain situations.

*Definition.** Two functions h_0 and h_1 are said to be *independently person-by-person monotonic (IPM)* iff $\forall\ z_1 \in Z_1$, $z_1' \in Z_1$ and $u_1' \in U_1$, if $h_1[\gamma_1^d(z_1), z_1] \le h_1(u_1', z_1)$ implies $h_1[\gamma_1^d(z_1), z_1'] < h_1(u_1', z_1')$, then $u_1' \ne \gamma_1^d(z_1')$. The function $\gamma_1^d(z_1)$ is here defined by $\gamma_1^d(z_1) = \arg\min_u h_0(z_1, u)$.

Suppose that for a given z_1, the desired solution is $\gamma_1^d(z_1)$. Consider now another $u_1' \in U_1$ where $h_1[\gamma_1^d(z_1), z_1] \le h_1(u_1', z_1)$. What IPM says is that if instead of z_1, the state of nature is z_1' and if the follower has $h_1[\gamma_1^d(z_1), z_1'] \le h_1(u_1', z_1')$, then u_1' will not be the desired solution for the leader when the state of nature is z_1'.

Theorem 1. (i) If identity (19) holds then (h_0, h_1) satisfies IPM. (ii) If (h_0, h_1) satisfies IPM, then

$$\arg\min_{u_1 \in U_1^d} h_1(z_1, u_1) \equiv \arg\min_{u_1 \in U_1^d} h_0(z_1, u_1) \quad \forall\ z_1 \quad (20)$$

where U_1^d is the range of $\gamma_1^d(z_1)$, and $U_1^d \subseteq U_1$. In particular, if $U_1^d = U_1$, then identity (19) holds.

Proof: [also reference Theorem 4.3.1 of Dasgupta, Hammond and Maskon (1979)].

(i) Suppose identity (19) holds. Let $\gamma_1^\dagger(z_1) \equiv \arg\min_{u_1} h_1(z_1, u_1) \cdot \forall\ z_1 \in Z_1$, $z_1' \in Z_1$ and $u_1' \in U_1$, if $h_1[\gamma_1^d(z_1), z_1] \le h_1(u_1', z_1)$ implies $h_1[\gamma_1^d(z_1), z_1'] < h_1(u_1', z_1')$, then we must have $u_1' \ne \gamma_1^d(z_1')$. For otherwise from identity (19), $\gamma_1^d(z_1') = \gamma_1^\dagger(z_1') = u_1'$, will contradict $h_1[\gamma_1^d(z_1), z_1'] < h_1(u_1', z_1')$.

(ii) Suppose (h_0, h_1) satisfies IPM. $\arg\min_{u_1 \in U_1^d} h_0(z_1, u_1) \equiv \gamma_1^d(z_1)$ by definition. Let $\hat{\gamma}(z_1) \equiv \arg\min_{u_1 \in U_1^d} h_1(z_1, u_1)$. If identity (20) does not hold, then there exists $z_1 \in Z_1$ such that with $u_1 = \gamma_1^d(z_1)$ and $u_1' \equiv \hat{\gamma}_1(z_1)$, $h_1(u_1', z_1) < h_1(u_1, z_1)$. On the other hand since $u_1' \in U_1^d$, there must exist $z_1' \in Z_1$ such that $\gamma_1^d(z_1') = u_1'$. Thus IPM implies that $u_1 \ne \gamma_1^d(z_1)$, a contradiction.

Remark: In certain situations it is easy to prove (h_0, h_1) not satisfying IPM by finding counter examples. If (h_0, h_1) does not satisfy IPM, then identity (19) will not hold. A sufficient and often convenient condition for (19) is, of course, to have $h_0 = h_1$ or $L_0(\gamma_0, \gamma_1, \xi) \equiv L_1(\gamma_0^d, \gamma_1, \xi)$, i.e. making the payoff function of the leader and the follower identical. But this is not necessary. IPM captures the essence of the requirement that the optimum of h_0 and h_1 are identical.

Example 3. [Example 1 of Ho, Luh and Muralidharan (1981).] Let

$$J_0 = E[L_0] = E[-\tfrac{1}{2}u_0^2 + u_0u_1 - u_1^2 + \xi_1u_0 + \xi_2u_1]$$

$$J_1 = E[L_1] = E[-2u_1^2 + (\xi_1 + \xi_2)u_1 + bu_0 - u_1]$$

where ξ_1 and ξ_2 are independent zero-mean Gaussian random variables and $b \ne 0$. Let the information structure be

$$\eta_0:\ u_1, \xi_1, \xi_2 \qquad \eta_1:\ \xi_2.$$

The team solution is $\gamma_0^d = u_1 + \xi_1$, $\gamma_1^d = \xi_2$. With $\gamma_0 = (1 - b + 3\xi_2)(u_1 - 1)/b + \xi_1 + u_1$, we have

*The concept of IPM naturally extends to the case of many followers. In fact this is the case in the economics literature where IPM was first introduced.

$$J_1 = E[-2u_1^2 + (\xi_1+\xi_2)u_1 + (1-b + 3\xi_2)(u_1 - 1) + b\xi_1 + bu_1 - u_1]$$
$$= E[-2u_1^2 + 4\xi_2 u_1 + \xi_1 u_1 + \text{terms not involving } u_1]$$

and

$$E/z_1[L_1] = E[-2u_1^2 + 4\xi_2 u_1 + \text{terms not involving } u_1].$$

Thus, $\arg\min_{u_1} E/z_1[L_1] = \xi_2$. On the other hand for $u_0 = \gamma_0^d = u_1 + \xi_1$, we have

$$J_0 = E[-\tfrac{1}{2}u_1^2 + (\xi_1+\xi_2)u_1]$$

and

$$E/z_1[L_0] = E[-\tfrac{1}{2}u_1^2 + \xi_2 u_1 + \text{terms not involving } u_1].$$

Thus $E/z_1[L_0]$ and $E/z_1[L_1]$ are obviously IPM since they satisfy identity (19).

4. STOCHASTIC INCENTIVE PROBLEMS II (NONNESTED CASE)

The only difference between problems treated in this Section and that of Section 3 is the fact that $\eta_1 \not\subseteq \eta_0$, i.e. the follower possesses private information not known to the leader. To simplify the discussion, let us assume that $z_1 = \xi$ and $z_0 = \phi$.* However, we encounter an immediate problem when we attempt to calculate the desired solution $\gamma_0^d(u_1)$ and $\gamma_1^d(z_1)$ using the analog of (16). We now face a dynamic team problem. Since $\eta_1 \not\subseteq \eta_0$, in general we do not know the optimal team solution. Nevertheless, it is possible to ask what should be the solution if the leader knows z_1. Denote this 'first best' solution, which is assumed to be unique, by $\gamma_0^t(u_0, z_1)$ and $\gamma_1^t(z_1)$, where the superscript t represents 'team'.† We define

(P-3). Find γ_0: $U^1 \to U^0$, $\gamma_0 \in \Gamma_0$ such that

$$\arg\min_{\gamma_1} E[L_1(\gamma_0(\gamma_1), \gamma_1, \xi)] \equiv \gamma_1^t(z_1) \quad (21a)$$

$$\gamma_0[\gamma_1^t(z_1)] \equiv \gamma_0^t[\gamma_1^t(z_1), z_1]. \quad (21b)$$

But (P-3) is generally infeasible. Unlike the nested case in Section 3, γ_0 is now only a single variable function. Definition (21b) completely specifies γ_0 leaving no degrees of freedom to satisfy identity (21a). To illustrate this, consider the following.

Example 4. The cost functions are

$$L_0 = u_0^2 + u_1^2 + u_0 u_1 + \xi_1 u_1$$
$$L_1 = 2u_0^2 + u_1^2 + 2u_0 u_1 + b\xi_1 u_1$$

where b is a constant known to the decision makers. The information structure is

$$\eta_0:\ u_1 \qquad \eta_1:\ \xi_1$$

ξ_1 is a zero-mean Gaussian random variable. The team solution, in which the leader also knows ξ_1, is

$$\gamma_0^t = \tfrac{1}{2}u_1, \qquad \gamma_1^t = -\tfrac{2}{3}\xi_1. \quad (22)$$

Condition (21b) now reads

$$\gamma_0[\gamma_1^t(\xi_1)] = \gamma_0(-\tfrac{2}{3}\xi_1) = \tfrac{1}{3}\xi_1$$

eliminating ξ_1 from the above identity, this requirement becomes

$$\gamma_0(u_1) = -\tfrac{1}{2}u_1. \quad (23)$$

Thus γ_0 is completely specified by (23) and no freedom is left to satisfy identity (21a). It is easy to verify that (21a) cannot be satisfied if γ_0 is given by (23) unless $b = \frac{2}{3}$. Thus the problem is not (linearly) i.c. for $b \neq \frac{2}{3}$. Note also that if this is not l.i.c., then it will not be i.c. (continuously or not) at all!

In order to make (P-3) feasible, additional restriction will be imposed on L_0 and/or L_1.

(A2). L_0 is independent of u_0.

Assumption (A2) not only eliminates identity (21b) completely but also makes unambiguous that $u_1 = \gamma_1^t(z_1) = \arg\min_{u_1} L_0(u_1, \xi)$ is the desired solution for the leader. From economics point of view, (A2) can also be justified as will be shown later in Section 5 when we discuss the allocation of public goods. Under (A2), (P-3) then becomes

(P-3'). Find γ_0: $U^1 \to U^0$ such that

$$\arg\min_{\gamma_1} E[L_1(\gamma_0(\gamma_1), \gamma_1, \xi)]$$
$$\equiv \arg\min_{\gamma_1} E[L_0(\gamma_1, \xi)]. \quad (24)$$

Again defining $E/z_1[L_0(u_1, \xi)] \equiv h_0(u_1, z_1)$ and $E/z_1[L_1(\gamma_0(u_1), u_1, \xi)] \equiv h_1(u_1, z_1)$, identity (24)

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*This is also the prevalent assumption in the economic literature. Under certain conditions, it is possible for the leader to induce the follower to reveal ξ truthfully under dominant strategy. In such case, the solution is independent of various distributional assumptions with respect to ξ.

†Any other desirable solution will be called 'second best'.

becomes

$$\arg\min_{u_1} h_1(u_1, z_1) = \arg\min_{u_1} h_0(u_1, z_1) \quad \forall\ z_1 \tag{25}$$

Theorem 2. (i) If identity (25) holds then (h_0, h_1) satisfies IPM. (ii) If (h_0, h_1) satisfies IPM, then

$$\arg\min_{u_1 \in U_1^d} h_1(u_1, z_1) \equiv \arg\min_{u_1 \in U_1^d} h_0(u_1, z_1) \quad \forall\, z_1 \tag{26}$$

where U_1^d is the range of γ_1^d, $U_1^d \subseteq U_1$. In particular, if $U_1^d = U_1$, then identity (25) holds.

Proof: exactly the same as that of Theorem 1.

In (P-3′), the follower's act u_1 determines L_0. However, suppose instead the leader controls not only u_0 but also u_1, and the sole function of the follower is to report the value of ξ, then the leader can achieve the desired solution provided the follower inform him the true value of ξ. Thus instead of (P-3′), we can consider the so-called *direct incentive problem*

(P-3″). Find $u_0 = \gamma_0(\gamma_1(\hat{\xi}))$ and $u_1 = \gamma(\hat{\xi})$ such that

$$\arg\min_{\hat{\xi}} E[L_1[\gamma_0(\gamma_1(\hat{\xi})],\quad \gamma_1(\hat{\xi}),\ \xi)] = \xi. \tag{27}$$

In (P-3″), the follower's decision $\hat{\xi}$ is simply to report the value of ξ which he alone knows. The leader's decision is to find γ_0 and γ_1 such that the follower reports truthfully. It is clear that if γ_0 solves (P-3′) then $g \equiv \gamma_0 \cdot \gamma_1'$ and γ_1' constitutes a solution of (P-3″),* i.e.

Theorem 3.† To every (P-3′) that admits a solution γ_0 there is an equivalent (P-3″) that has a solution $u_0 = \gamma_0[\gamma_1'(\hat{\xi})]$, $u_1 = \gamma_1'(\hat{\xi})$ and $\hat{\xi} = \zeta$.

Example 5. The cost functions are

$$L_0 = \tfrac{1}{2} u_1^2 + u_1\xi$$
$$L_1 = u_0 + \tfrac{1}{2} u_1^2 + 2bu_1\xi$$

where b is a positive constant known to the decision makers. The information structure is

$$\eta_0\colon u_1 \qquad \eta_1\colon \xi$$

where ξ is a random variable. If the leader announces $u_0 = (b - \tfrac{1}{2})u_1^2$, then $L_1 \equiv 2bL_0$ and he will obtain his team solution. This indirect incentive problem can be converted into a direct one by defining

$$u_0 = (b - \tfrac{1}{2})\hat{\xi}^2, \qquad u_1 = -\hat{\xi}$$

where $\hat{\xi}$ is the follower's reported value of ξ. This direct incentive problem is cheat proof in the sense that the best thing the follower can do is to report the truth.

5. RELATIONSHIP TO ECONOMIC LITERATURE

5.1. *Relation with social choice theory*

In this subsection various concepts of the social choice theory will be introduced and their relations with previous sections will be indicated. In economics, social choice theory deals with incentives and particularly with incentives to the correct revelation of private information for public use. Though such problems were known to exist for a long time, social choice theory received its main impetus when Arrow formulated his famous impossibility theorem (Arrow, 1951).

In social choice theory it is assumed that there is a finite number (m) of agents, followers in our terminology; and a set A of alternatives, of which a typical member is indicated by a. In the context of this paper, $a = (u_0, u_1, \ldots, u_m)$. For the sake of simplicity, we shall assume

(A3). Each agent i ($i = 1, \ldots, m$) has a *strict* ordering P_i over A. Where an ordering is a complete, transitive, asymmetric, binary relation. Note that many results are also valid for indifferences.‡

Let $\Sigma(A)$ denote the class of admissible orderings, then $P \equiv (P_1, \ldots, P_m) \in [\Sigma(A)]^m$ is called a preference profile. Let $S(A)$ be the class of allowed social orderings on A. A social welfare function (SWF) is a mapping from $[\Sigma(A)]^m$ into $S(A)$; it will be indicated by $\mathscr{L}_0$§ The SWF assigns a social ordering to any allowed preference profile.

We will mention four possible properties of

*There may be other solutions to (P-3″). But we assume that if 'truth' is one of the solution then it will be chosen, see Dasgupta, Hammond and Maskin (1979).

†Reference Theorem 4.1.1 of Dasgupta, Hammond and Maskin (1979) for the multi-follower case.

‡In previous sections we started with cost functions $L_i(u_0, u_1, \ldots, u_m)$, where the contours are curves of indifference. In general, a preference ordering over A is more basic, since there is no topological structure as there is in the case when preferences are given through cost functions.

§In terms of notations of previous sections, $\mathscr{L}_0$ maps the followers' payoff functions (individual preference orderings) $L_1, \ldots, L_m$ into the leader's payoff function (social ordering) L_0. However in social choice theory $\mathscr{L}_0$ in general is not given. Instead one tries to construct a $\mathscr{L}_0$ having several desirable properties. The impossibility theorems in social choice theory are important to us in the sense that they state that no $\mathscr{L}_0$ exists which satisfies certain reasonable properties.

SWF. An SWF satisfies the assumption of *universal domain* (UD) if $\Sigma(A)$ is the class of all possible orderings on A. An SWF satisfies the *pareto optimality* (PO) if all agents prefer a_1 to a_2, then the social ordering also prefers a_1 to a_2; formally: $a_1P_ia_2\ \forall\ i$ implies $a_1\mathscr{L}_0(P)a_2$. An SWF satisfies the *independence of irrelevant alternatives* (IIA) if the ranking of two alternatives by $\mathscr{L}_0(P)$ only depends on the ranking of these alternatives by the agents. That is, for arbitrary a_1, a_2 and P_i, P'_i such that, $a_1P_ia_2$ if and only if $a_1P'_ia_2$, then $a_1\mathscr{L}_0(P)a_2$ if and only if $a_1\mathscr{L}_0(P')a_2$. In other words, it should not matter how many other alternatives are between the two. 'Intensities' (a_1 is a little bit preferred to a_2 or a_1 is by far preferred to a_2) can not be taken into account if IIA holds. Lastly, an SWF is called *dictatorial* if there exists an agent i whose P_i determines the social ordering $\mathscr{L}_0(P)$.

Arrow's impossibility theorem states that any SWF which satisfies UD, PO and IIA must be dictatorial, provided that set A contains at least three elements.

From the decision making's point of view, SWF is just an intermediary step used to define an optimal social alternative from A. This leads naturally to the concept of *social choice function*. Given the preferences of the agents, an SCR assigns an alternative to any allowed preference profile P, i.e. it maps $[\Sigma(A)]^m$ into A. SCR corresponds to the function $\gamma^t(\cdot)$ of Section 4 or $\gamma^d(\cdot)$ of Section 2. For SCR there is also an impossibility theorem which will be formulated after having introduced a number of SCR properties.

An SCR $[\gamma(\cdot)]$ satisfies *citizen sovereignty* (CS) if for every $a \in A$ there exists a preference profile P such that $\gamma(P) = a$. For direct incentive problems, an SCR is *manipulatable* means that some agents can be better off by not reporting their true preferences (they lie instead). An SCR is *truthfully implementable in dominant strategies* (TID) if there is no preference profile at which it is manipulatable. This property is sometimes called 'strongly individually incentive compatibility'. An SCR is *dictatorial* if there is an agent i such that for any $P \in [\Sigma(A)]^m$ his reported alternative is always the social choice.

In the spirit of Theorem 3, we can see that to every dominant strategy there exists an equivalent direct strategy satisfying TID. Thus from now on we shall consider direct incentive problems only, with SCR corresponding to $\gamma_0(\cdot)$. The Gibbard–Satterthwaite impossibility theorem (Gibbard, 1973; Satterthwaite, 1975) then states that if the range of an SCR has at least three alternatives, and this SCR satisfies CS, TID and UD, it is dictatorial. We introduce the leader to be the person who chooses $\gamma_0(\cdot)$, then a possible goal of the leader is to choose γ_0 in such a way it satisfies the reasonable assumptions of being cheat proof (i.e. TID), nondictatorial, UD and CS. The Gibbard–Satterthwaite impossibility theorem then says that such a choice does not exist. Note that such a γ_0 does not exist, in spite of the fact that no $\gamma_0'(\cdot)$ was given at the outset of the problem as was the case in Section 4. Thus the result of the Gibbard–Satterthwaite impossibility theorem is even stronger. Also note that currently there are preference profiles, whereas in Section 4 the preference profiles already had a lot of structures, the only unknowns being the 'state of nature' ξ, which were random variables.

In search for positive results, various suggestions can be made, such as

(a) restrictions on the domain of preferences;

(b) introduction of a mixed (random) social choice strategy;

(c) weaken the requirement of dominant strategy.

Various other possibilities for positive results have been mentioned in Green and Laffont (1979).

5.2. *Positive and negative results*

In this subsection we shall give two classes of problems which can be truthfully implemented in dominant strategies by restricting the domains of preferences (Examples 6 and 7) and one class for which this is not possible (Example 8). The last example in this subsection deals with a problem which is truthfully implementable with mixed γ_0, however, not through any pure γ_0 (Example 9).

Example 6. We are given m followers of whom the payoff functions (to be maximized) are

$$L_i = v_i(u_0) + u_{0i}, \qquad i = 1, \ldots, m. \tag{28}$$

The leader chooses both u_0 and u_{0i}, $i = 1, \ldots, m$, and he wants to maximize

$$L_0 = \Sigma v_i(u_0). \tag{29}$$

Note that the decisions u_{0i}, $i = 1, \ldots, m$ do not enter in the leader's criterion, which is in agreement with assumption (A2) in Section 4. The decision u_0 is usually thought of as a public good, which affects all the followers; u_{0i} is thought of as a personal reward (or penalty if it is negative). The leader does not know the (utility) functions $v_i(\cdot)$.* Each follower reports a function $\hat{v}_i(\cdot)$, not necessarily the true one, to the leader. Let the leader base u_0 and u_{0i} on the reported $\hat{v}_i$ in the following way

$$u_0 = \arg\min_{u_0} [\Sigma \hat{v}_i(u_0)] \quad (30a)$$

$$u_{0i} = \sum_{j \neq i} \hat{v}_j(u_0). \quad (30b)$$

Now it easily follows that it is in the interest of each follower to report the true $v_i(\cdot)$, independent of what the other followers will do (truth or not). Thus the problem can be truthfully implemented in dominant strategies.

This problem has been extensively studied by Groves (1973) and (30a), (30b) is known as the Groves mechanism.

Example 7. In this example we assume the followers' payoff functions are single peaked. These payoff functions are given by $L_i(u_0)$, $i = 1,\ldots, m$. For the sake of simplicity we assume u_0 to be a scalar [for extensions see Sen (1970)]. In this case, the function L_i being single peaked is identical to L_i being unimodal. The leader decides u_0 based upon the reports from the followers regarding the optimal u_0^i that maximizes $L_i(u_0)$. That is, the leader not knowing $L_i(\cdot)$, $i = 1,\ldots, m$, bases his u_0 on the reported $\hat{u}_0^i$. The leader does not have a payoff function himself; he simply wants to construct a direct incentive mechanism $u_0 = \gamma_0(\hat{u}_0^1,\ldots, \hat{u}_0^m)$ which has many desirable properties. It easily follows that if u_0 is chosen according to the 'median voter rule', i.e. he chooses u_0 such that there are as many $\hat{u}_0^i$ to the left of him as there are $\hat{u}_0^j$ to the right, then this SCR is TID, PO and nondictatorial. The verification of this explanatory statement is left as an exercise for the reader.

The reason why the Gibbard–Satterthwaite impossibility theorem does not apply to Examples 6 and 7 is that the assumption UD is violated. Both functions (28) and the single peaked payoff functions are small subclasses of all preference orderings.

The following example (Dasgupta, Hammond and Maskin, 1979) resembles the previous one in certain aspects, but the result is negative: no 'reasonable' direct mechanism exists, even though the assumption on UD is also not satisfied.

*This lack of knowledge may sometimes be characterized alternatively by a random variable ξ_i with $v_i(u_0, \xi_i)$. Then Example 6 essentially becomes the (P-3″) version of Section 4.

†In this example we shall ignore the 'strict ordering' part of (A3), and allow indifferences among alternatives.

Example 8. Consider an economy with n goods and m decision makers, all being followers. We assume $n \geq 2$, $m \geq 2$. There are fixed positive stocks $(w_1,\ldots, w_n)$ of these n goods and the task of the leader is to device a partitioning of the available goods among the followers. We assume that the payoff function of each follower is given by a strictly convex and strictly monotonic function†

$$L_i(u_{0i}), \quad i = 1,\ldots, m$$

where $u_{0i} = (u_{0i1},\ldots, u_{0in})$ is follower i's share of the goods. Each follower is interested in maximizing his own payoff function. $u_{01},\ldots, u_{0m}$ will be chosen by the leader, after the followers have revealed their payoff functions $\hat{L}_i$; i.e. $u_0 \equiv (u_{01},\ldots, u_{0m}) = \gamma_0[\hat{L}_1(\cdot),\ldots, \hat{L}_m(\cdot)]$. The reported $\hat{L}_i(\cdot)$ is of course not necessarily the true $L_i(\cdot)$.

Thus formulated, the leader faces a direct incentive problem. His goal is to distribute the goods in such a way that the pareto optimality condition is satisfied. (Note that the leader does not have his own payoff function. We could think of $\Sigma\lambda_i L_i$ for different combinations of nonnegative λ_is as the leader's payoff and require the social choice to be pareto optimal.) The function γ_0 satisfies the pareto optimality if $L_i(u_0) \geq L_i(u_0') \ \forall \ i$, with at least one strict inequality sign, then $u_0' \neq \gamma_0(L_1,\ldots, L_m)$. Theorem 4.4.1 of Dasgupta, Hammond and Maskin (1979) now states that if γ_0 yields a pareto optimal outcome and if γ_0 can be truthfully implemented in dominant strategies, then γ_0 must be dictatorial. A dictatorial mechanism γ_0 means here that one of the followers gets all the goods. Though this is a pareto solution, it is clear that it is a very unsatisfactory one. The proof of this result can be found in Dasgupta, Hammond and Maskin (1979).

We will conclude this subsection with an example in which a mixed γ_0 is truthfully implementable. Random social choice procedures have been suggested before (Green and Laffont, 1979), but in a different context. As a possible procedure for a direct incentive scheme it was suggested in Green and Laffont (1979) to choose a follower at random and using his announced preferences as the social ordering. Example 9, which deals with an indirect incentive, is different in nature.

Example 9 (*mixed strategies*). Let

$$L_0 = u_0^2 + u_1^2$$

$$L_1 = (u_0 - \xi)^2 + (u_1 + 1)^2$$

with the following information structure

$$\eta_0: u_1 \qquad \eta_1: \xi$$

where ξ is a zero-mean Gaussian random variable with unit covariance. By depicting the contours for the possible L_1 cost function, it is easily shown that any continuous $u_0 = \gamma_0(u_1)$ will not lead to $L_0 = 0$. Suppose that the leader announces

$$u_0 = 0 \text{ for } u_1 \geq 0 \text{ and } u_0 = -Nu_1 \text{ for } u_1 < 0$$

$$u_0 = 0 \text{ for } u_1 \geq 0 \text{ and } u_0 = +Nu_1 \text{ for } u_1 < 0$$

both with chance $\frac{1}{2}$. Here $N > 1$ is a constant. It is easily shown that the best the follower can do is to choose $u_1 = 0$; another choice of u_1 will lead to higher average costs (higher than $L_1 = 2$) for the follower. Therefore the problem is i.c. in mixed strategies.

5.3. *Other related problems*

So far we have mainly dealt with problems in which the leader was able to obtain his team solution. However, what can the leader achieve if he knows that the team solution is beyond reach? Consider Example 9 and let us restrict this example to pure strategies. The leader can not obtain his team solution, though he can get arbitrarily close. Other problems can be devised where he can not ever get ϵ close (consider Example 5 with $b < 0$. The leader can not obtain his team payoff by the method described above and probably can not achieve it at all.) In its generality, this problem has not (yet) been solved. Only a few special problems have been successfully treated (Mirrlees, 1971; Shavell, 1979). In Tolwinski (1980) some results in this vein are given for the deterministic case.

Let us give a brief description of a problem considered in Spence (1977). An individual's preference is characterized by two parameters, the marginal valuation of income, denoted by λ, and the valuation of the good relation to income, denoted by ϕ, The parameters λ and ϕ are jointly distributed in the population according to the distribution $h(\lambda, \phi)$. If an individual i with parameters λ and ϕ buys u_i units of the good for $\gamma_0(u_i)$ dollars, the social gain is

$$L_i = \lambda[v(u_i, \phi) - \gamma_0(u_i)]$$

where the function $v(u_i, \phi)$ can be thought of as the dollar value of u_i units of good to this individual. The leader, who knows $h(\lambda, \phi)$, but does not know the λ, ϕ values for each individual, wants to choose $\gamma_0(\cdot)$ in such a way to maximize

$$L_0 = \iint \lambda[v(u^*, \phi) - \gamma_0(u^*)]\ h(\lambda, \phi)\, d\lambda\, d\phi$$

where u^* is determined by maximization of L_i, expressed in terms of λ and ϕ.

The solution to this problem has been obtained in Spence (1977) given that the function $v(u_i, \phi)$ satisfies some reasonable differentiability requirements. Crucial in the determination of the solution is that u_i is a scalar (calculus of variations techniques are used in the derivation, in which u_i is treated as an independent variable).

Another related problem is the one of the principal and agent, considered in Shavell (1979). The principal is the leader and his payoff is given by $L_0[x(u_1, \xi) - u_0(x)]$, where x is a function of u_1, the agent's decision, and ξ, a random variable; u_0 is the payment of the principal to the agent. The agent's payoff is $L_1(u_0, u_1)$. As a specific application, one can think of u_0 as the fee which the principal (client) pays his agent (lawyer); u_1 is the lawyer's effort in the court case, of which the outcome is $x(u_1, \xi)$. The principal and agent are each assumed to act so as to maximize their expected utilities

$$J_0(\gamma_0, u_1) = \int L_0[x(u_1, \xi) - \gamma_0(x(u_1, \xi))]p(\xi)\, d\xi$$

$$J_1(\gamma_0, u_1) = \int L_1[\gamma_0(x(u_1, \xi)), u_1]p(\xi)\, d\xi$$

where $p(\xi)$ is the density function of ξ. The major difference between this problem and other problems considered before is that u_1 is not directly observable by the principal. The effort of the agent is seen through the function $x(u_1, \xi)$ (corrupted measurement). The discussion in Shavell (1979) centers around the question of how the fee $u_0 = \gamma_0(x)$ is related to the outcome x and, in addition, what happens if the principal would have some information (which may be imperfect) about the agent's effort u_1 directly.

6. CONCLUSION

A unification of some recent results in Stackelberg problems has been provided. A link with incentive problems in the economic literature has been made. Of all incentive problems and corresponding solutions, only a very small portion, like the tip of an iceberg, has been scratched.

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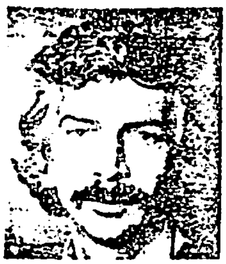

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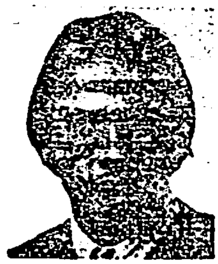

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