Short Papers

Impact of Storage on Load Management by Utilities

RAMAL MURALIDHARAN, MEMBER, IEEE, YU-CHI HO, FELLOW, IEEE, AND PETER B. LUH

Abstract—In this paper we study the impact of storage on load management by electric utilities. The results we have obtained are based on a simple model with linear storage costs and quadratic generating costs. Inefficiency of storage is a parameter in the model. An optimal control problem, with state variable and control variable inequality constraints, is formulated. It is shown that the optimal storage/retrieval strategies may be summarized in terms of a "peak line," above which the load curve will be shaved off and a "valley line," below which the load curve will be filled. The control strategy, albeit based on a simple model, makes explicit the role of storage and generation parameters in the planning for the operation of the system.

I. Introduction

The electric utilities[1] in the United States face the peak load problem. This problem arises because the utility must meet the demands due to a fluctuating load instantaneously, thus incurring idle capacity cost, while capacity expansions require a fairly long gestation period. Most utilities use additional peaking generators (which have low capital cost to offset their idle time but high running cost) to satisfy the excess loads during the short duration peak periods. The unit cost of electricity production during the peak periods is thus higher than that in the off-peak periods. The producers (utilities) may pass on the increased cost to the consumer by some peak-load pricing schemes using various criteria such as maximizing social welfare, etc. [1]–[3].

In this paper we investigate how the use of storage may help a utility to offset some of the problems associated with peaking loads. In particular, we investigate how storage may be used as a load management tool —to shape the load curve by filling valleys and shaving off peaks. Our purpose here is primarily to study the qualitative aspect of the interaction of storage with generation dispatch schedules. To this end, we assume a simple deterministic and aggregated model for storage and generation. We are fully cognizant of the many complicating conditions that obtain in the real world, e.g., the random nature of the load, the forced outage of generators, and the inherent discreteness and distributive nature of generation and load in a power system [4]. Ignoring these complications enables us to obtain an explicit solution, and, thus, make precise the role of certain parameters in the problem. Before actual implementation, these considerations must no doubt be incorporated, guided by the insights that we have developed here. In this sense, our results may be more useful in the planning (as opposed to operational) stage of the system.

The study of combined storage and generation is not new. McDaniel and Gabrielle [5] have described a specific scheduling problem involving pumped hydro storage in combination with steam generating capacity. Many practical complications are included in their treatment. Their strategy is to retrieve whenever the system lambda exceeds the incremental cost (assumed to be a constant) of the pumped storage hydro. Only the retrieval part of the operation is considered, and no optimality in a strict sense is claimed. Cobian [6] deals with another specific problem of obtaining the optimal scheduling of a pumped storage hydroelectric plant in combination with several interconnected power systems. Numerical solutions are obtained by the dynamic programming method. The contribution of our paper is to make explicit the influence of the cost and efficiency of storage in the generation dispatching schedule, i.e., when to store and retrieve as a function of system parameters. This is accomplished via the (nonnumerical) solution of an optimal control problem subject to state and control variable inequality constraints.

Manuscript received April 2, 1979; revised September 10, 1979, and February 25, 1980. Paper recommended by J. D. Glover, Chairman of the Energy Systems Committee. This work was supported by the Department of Energy, U.S. Government, under Contract ET-78-C-01-3252.

R. Muralidharan is with Bolt Beranek and Newman Inc., Cambridge, MA 02138.

Y.-C. Ho and P. B. Luh are with the Division of Applied Sciences, Harvard University, Cambridge, MA 02138.

[1]The problem addressed in this paper is common to many other public enterprises as well, but we shall focus the discussion on electric utilities to keep the terminology simple and unambiguous.

II. Problem Formulation

A. The Load Curve

The utility is required to cater to an aggregate load due to a large number of customers. Fig. 1 is an example of a typical daily load curve (a plot of the kilowatt demand versus the time of the demand) for a large eastern summer peaking utility. Although the exact load curve is unpredictable as it will also depend on the random elements such as weather, etc., the average load does fluctuate in a predictable way. The example load curve of Fig. 1 has a peak over the late afternoon hours and a valley over the early morning hours. In our model, the load curve $q(t)$ is assumed to be known for the planning period $[0,T]$, which may be a day, a week, or a year, etc. It is assumed to be deterministic, piecewise continuous, and periodic with $q(0)=q(T)$ by definition.

B. Cost of Generation

The cost function for electricity production has received considerable attention in recent research. Widely adopted is a multitechnology model where each technology would incur a capacity cost and a linear operating cost. The system running cost is then a piecewise linear function. We shall adopt a simple quadratic function to reflect the cost of generation. Such a quadratic function may be viewed as an approximation to the piecewise linear cost for the multitechnology model. Note that utilities do use a quadratic cost function in other contexts [8]. The generation cost function is assumed to be

$$GC=\int_0^T\left(\frac{1}{2}a_2g^2+a_1g+a_0\right)dt \tag{1}$$

where $g(t)$ is the actual generation level (kilowatts) at time t, and a_i are parameters[2] assumed to be known.

C. Cost of Storage

It is assumed that some facility for storage is available to the utility. Storage may be of different types: for example, batteries, compressed gas, or pumped hydro [9]. Irrespective of its type, we shall model storage in terms of a "round trip" efficiency factor e and a storage cost SC. The efficiency factor $e(0<e<1)$ denotes the fact that when 1 kWh of electricity is stored only e kWh can be retrieved to meet later demand. This affects the load flow equation (4) below. The storage cost SC is assumed to be of the following form:

[2]We shall take a_i to be constant over time to keep our discussion simple. Note that $a_0>0$ as it reflects the capacity cost. The marginal running cost (a_2g+a_1) is positive and increasing (linearly) with g, as less efficient generation is brought into service. We also assume that $a_2>0$.

0018-9286/80/0800-0685$00.75

$$SC=\int_0^T (b_1x+b_0)\,dt \tag{2}$$

where $x(t)$ denotes the amount of electricity in storage at time t.[3] The linear term in the integrand of (2) can be interpreted in different ways. For example, it may reflect in average terms the increasing cost of storage capacity, or it may represent the operating cost of retrieving the stored energy.

D. The Optimization Problem

The problem facing the utility is to determine the best way to meet the load demand using available generation and storage facilities. We shall next formulate this problem as an optimal control problem. It will be assumed that the utility is interested in determining a storage dispatch strategy to minimize the total cost of generation and storage. The dynamic equation governing the stored quantity $x(t)$ (in kilowatt hours) is

$$\dot{x}(t)=u(t)-v(t) \tag{3}$$

where $u(t)$ is the rate of storage and $v(t)$ the rate of retrieval. The generation $g(t)$ together with the amount of retrievable from storage[4] $ev(t)$ must be sufficient to meet the aggregate demand $q(t)$ from the consumers, in addition to providing for storage at the rate $u(t)$. That is

$$g(t)+ev(t)=u(t)+q(t). \tag{4}$$

Important constraints on the storage variables that will be included in our model are[5]

$$u(t)\geq 0 \tag{5}$$

$$v(t)\geq 0 \tag{6}$$

$$x(t)\geq 0. \tag{7}$$

In the present model, the capacity constraints on $g(t)$ and $x(t)$ and the nonnegativity constraint on $g(t)$ will be neglected.[6]

The total cost of generation and storage is obtained by summing (1) and (2). Using (4) this total cost is

$$J=\int_0^T\left[\frac{1}{2}a_2(q+u-ev)^2+a_1(q+u-ev)+a_0+b_1x+b_0\right]dt. \tag{8}$$

Note that J will depend on the state $x(0)$ of initial storage as well as the storage and retrieval functions.

The optimization problem facing the utility is then as follows: determine the storage function $u(\cdot)$, retrieval function $v(\cdot)$, and the level of initial storage $x(0)$[7] so as to minimize the total cost of J in (8). The constraints are the dynamic state equation (3), the inequality constraints on the control variables (5) and (6), and the state variable inequality constraint (SVIC) (7). In addition, we shall require that $x(T)=x(0)$ so that the same control functions will continue to be optimal for the same load curve over successive periods of time. In view of (3), this periodicity constraint translates to the "isoperimetric" constraint

$$\int_0^T (u(t)-v(t))\,dt=0. \tag{9}$$

This optimization problem is a standard one in optimal control theory (see [10], ch. 3). The optimality conditions are summarized in the next section.

E. Optimality Conditions

It is convenient to define the Hamiltonian function:

$$H(t)=\frac{1}{2}a_2(q(t)+u(t)-ev(t))^2+a_1(q(t)+u(t)-ev(t))+a_0 + b_1x(t)+b_0+(\lambda(t)-\mu(t)+\eta)(u(t)-v(t)).$$

Note that H is a function of the storage (state) variable $x(t)$, the storage rate (control) variables $u(t)$ and $v(t)$, and the Lagrange multipliers $\lambda(t)$, $\mu(t)$, and η. The multiplier function $\lambda(t)$ accounts for the dynamic constraint (3), the multiplier function $\mu(t)$ accounts for the SVIC (7), and the constant multiplier η accounts for the isoperimetric constraint (9).

The optimality conditions are as follows:

$$u(t)H_u=0,\quad u(t)\geq 0,\quad H_u\geq 0 \tag{10}$$

$$v(t)H_v=0,\quad v(t)\geq 0,\quad H_v\geq 0 \tag{11}$$

$$\mu(t)x(t)=0,\quad \mu(t)\geq 0,\quad x(t)\geq 0 \tag{12a}$$

$$\dot{x}(t)=u(t)-v(t),\quad \dot{x}(t)=0\ \text{ on the boundary} \tag{12b}$$

$$\dot{\lambda}(t)=-H_x,\quad \lambda(0)=0,\quad \lambda(T)=0. \tag{13a}$$

$\lambda(t)$ is discontinuous at the "entry point" of the SVIC boundary where $x(t)$ just becomes zero. (13b)

$$\int_0^T (u(t)-v(t))\,dt=0. \tag{14}$$

From (13) $\lambda(t)$ is a piecewise continuous linear function of the form $\lambda(t)=b(k-t)$, where the constant k will be determined by the discontinuities in λ. In particular, the sum of all jumps in $\lambda(t)$ equals bT.

III. Results

In this section we analyze the implications of the optimality conditions stated in Section II-E.

A. Implications of Storage Inefficiency

If storage is not perfectly efficient (that is, $e<1$), then it is not optimal to have simultaneous storage and retrieval. This intuitively obvious result, which prohibits "redundant flow in the generator-storage loop," is[8]

Proposition 1: If $e<1$ then $u(t)v(t)=0$.

Proof: Let $u(t)>0$. Then from (10) $H_u=0$, which together with $e<1$ yields $H_v>0$, and hence $v(t)=0$ by (11). Similarly, we can prove that $u(t)=0$ whenever $v(t)>0$.

B. Implications of Nonzero Storage Cost

If storage is not free, then it appears intuitively clear that the optimal storage strategy must be such as to prohibit "residual storage" over the entire period $[0,T]$. That is, the storage must be in an empty state at some instant during the period $[0,T]$. This is:

Proposition 2: If $b_1>0$ then $x(t)=0$ for some t in $[0,T]$.

Proof: By way of contradiction let $x(t)>0$ for all t in $[0,T]$ so that $x(t)$ never enters the boundary of the inequality constraint (7). The absence of entry points and (13b) imply that $\lambda(t)$ is continuous. Since $H_x=b_1>0$, there exists no λ which can satisfy (13a). Thus, we have a contradiction and the proof is complete.

Since the criterion J is strictly convex and all constraints and dynamics are linear, the solution to the optimal control problem facing the utility is unique [11] when $e<1$ and $b_1>0$. When $e=1$ and/or $b_1=0$ the solution is no longer unique. However, it can be shown that there exists a solution which has the desirable properties of "no redundant flow" and "no residual storage." Henceforth, we shall consider only such a solution which would satisfy the following conditions:

[3] A more general form of (2) is treated in Section IV later. The b_i here are assumed to be nonnegative constants.

[4] For convenience we have introduced a storage rate $u(t)$ and a retrieval rate $v(t)$. Since the storage cost is linear in $x(t)$, we have lumped the efficiency factor e with $v(t)$ (see also Section IV for a more general model).

[5] In Section IV we shall incorporate maximum power flow constraints on u and v.

[6] This is done to simplify our derivations and to focus our attention on other things we consider to be more important. Note that, if sufficient generating and storage capacities are available, then the capacity constraints can be ignored. Also, since we expect $v(t)$ to be small compared to $q(t)$, (4) would imply that $g(t)$ would indeed be positive. Note that neglecting the capacity constraints makes the capacity cost parameters a_0, b_0 irrelevant in our problem formulation. However, capacity constraint on generation can always be studied parametrically using the solution to the procedure posed here.

[7] The parameter $x(0)$ influences the time at which $x(t)$ reaches the constraint boundary (7), as well as the duration for which $x(t)$ stays on this boundary. Thus, $x(0)$ is a legitimate optimization parameter. If desired, $x(0)$ may be chosen to keep $x(t)$ above some specified positive value [instead of zero as in (7)], to allow for security against random unexpected changes in the demand level $q(t)$.

[8] Of course, in practice, other considerations such as "load following" may dictate redundant flow.

Fig. 1. Typical daily load curve for a large eastern summer peaking utility (adapted from [3]).

$$u(t)v(t)=0 \quad \text{for all } t \text{ in } [0,T] \tag{15}$$

$$x(t)=0 \quad \text{for some } t \text{ in } [0,T]. \tag{16}$$

C. Control Strategy

Recall that intuitively the problem facing the utility was to determine the optimal storage strategy to shave off system load peaks and fill system load valleys. In this section we shall show that the optimal storage/retrieval within each cycle of storage operation indeed has this simple character.

In terms of the peak and valley lines, to be defined below, the control of storage can be simply stated as follows:[9]

1) store whenever the load falls below the valley line and let generation follow the valley line;

2) beyond the exit point (where $x(t)$ just leaves the SVIC boundary), retrieve whenever the load rises above the peak line and let generation follow the peak line.

3) When load curve is above the valley line but below the peak line, let generation follow the load curve and no storage or retrieval is to be done.

The derivation of this control strategy is based on two propositions which we shall state below after establishing some notations and definitions keyed to Fig. 2. Proofs are given in the Appendix.

By Proposition 2, there exists at least one entry point in each planning period. If t_1 is an entry point, then t_1+T is also an entry point. Thus the interval $[0,2T)$ always contains at least a pair of entry points. Let t_1, t_3 denote any consecutive entry points,[10] with $0 \leq t_1 < t_3 < 2T$. Then we define the storage cycle as the interval $(t_1, t_3]$ between two consecutive entry points. Let t_2, with $t_1 \leq t_2 < t_3$, be the exit point. Over the storage cycle $(t_1, t_3]$, define valley line as the straight line with slope b_1/a_2 and intercept $c_1 = -(b_1 k - b_1 t_1 + \eta + a_1)/a_2$ at $t=t_1$, where the constant k accounts for the jumps in $\lambda(t)$ [recall footnote keyed to (13b)]. Define peak line as the straight line with slope $b_1/(ea_2)$ and intercept $c_2 = -(b_1 k - b_1 t_1 + \eta + ea_1)/(ea_2)$ at $t=t_1$.

Proposition 3: Over the storage cycle $(t_1, t_3]$ let t_2, with $t_1 \leq t_2 < t_3$, be the first instant at which the load curve crosses the valley line from below.[11]

[9] Our result can be related to the solution in terms of marginal cost obtained in [5]. We shall do this for a specialized case in Section IV.

[10] That is, there is no other entry point between t_1 and t_3.

[11] In view of this proposition, t_2 as defined is indeed the exit point on the SVIC boundary.

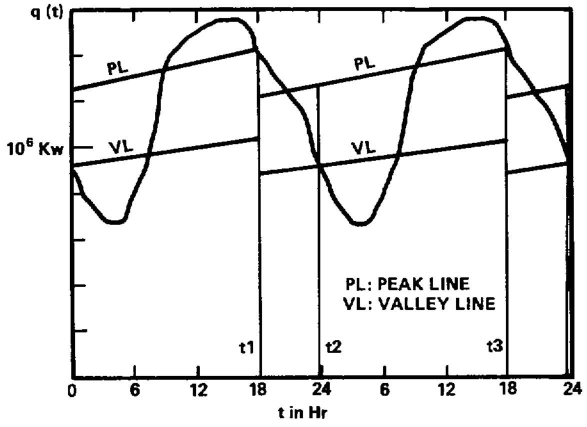

Fig. 2. Peak and valley lines for a typical daily load curve.

1) In $(t_1, t_3]$, it is optimal to store whenever the load curve falls below the valley line and to let generation follow the valley line.

2) In $(t_2, t_3]$, it is optimal to retrieve whenever the load curve rises above the peak line, and to let generation follow the peak line.

3) The load curve $q(t)$ crosses the peak line at t_3.

Once the entry points t_1, t_3 are known, the above propositions may be used to obtain the optimal control strategy in a straightforward manner. The only remaining task now is to locate the entry points and the intersection points where $q(t)$ crosses the peak and valley lines. This can be done with the help of

Proposition 4: If the load curve $q(\cdot)$ is continuous for some t then the optimal storage function $u(\cdot)$ and retrieval function $v(\cdot)$ are also continuous at t.

D. Algorithm for Determining the Optimal Control Strategy

In this section we shall state the algorithm for the most prevalent case where there is only a single entry point in $[0,T]$.

Step 1: Guess a value for the multiplier η associated with the isoperimetric constraint (9). Draw a straight line L with slope b_1/ea_2 and intercept $-(\eta+ea_1)/ea_2$ at $t=0$.

Step 2: Determine the point t_1 where $q(t)$ crosses L from above. Then t_1 is a candidate entry point and the interval (t_1, t_1+T) is a candidate storage cycle. (If t_1 is nonunique, then each of them must be considered as a candidate entry point.)

Step 3: In $(t_1, t_1+T]$, draw peak and valley lines and determine $u(t)$ and $v(t)$ according to Proposition 3. Since there is only one entry point in $[0,T)$, $\lambda(t) = b(k-t) = b(T-t)$ beyond the entry point.

Step 4: Adjust η such that the state variable nonnegativity constraint (7) and the isoperimetric constraint (9) are satisfied.

The above algorithm is illustrated for a simple example in the next section.

E. Example

Consider a typical eastern summer peaking utility faced with the problem of satisfying an aggregate demand with load curve as shown in Fig. 1. Let the efficiency and cost parameters for the utility be as follows:

$$e=0.73$$

$$a_2 = 6\times 10^{-8} \text{ (\$/kW·kWh)}$$

$$b_1 = 4\times 10^{-4} \text{ (\$/kW·kWh)}$$

$$a_1 = a_0 = b_0 = 0.$$

Note that a_2 was chosen so that the average cost when $q(t) = 10^6$ kW is 3 cents/kWh.

For the above parameter values, the slope of the valley line is 0.67×10^{-4} (kW/h) and the slope of the peak line is 0.91×10^{-4} (kW/h). The optimal storage and retrieval strategies obtained by the algorithm described in Section III-D are illustrated in Fig. 3. It is seen that the optimal control strategy is as follows:

1) store during the hours of (roughly) midnight until 7 AM. The generation will follow the valley line during this period;

2) no storage or retrieval to be done from 7 AM until about 9 AM and again from 6 PM until about midnight;

3) retrieve during the hours of 9 AM until about 6 PM, the generation will follow the peak line during this period.

IV. Some Generalizations

The model described in Section II can be generalized to include the following additional features without changing the essence of the results:

1) a term proportional to the retrieval rate $v(t)$ is added to the storage cost (2), i.e.,

$$SC=\int_0^T (b_2 v+b_1 x+b_0)\,dt \tag{2'}$$

with

$$b_2 \geq 0;$$

2) maximum power flow constraints for both storage and retrieval are included, i.e.,

$$0 \leq u \leq u_m \tag{5'}$$

$$0 \leq v \leq v_m \tag{6'}$$

with

$$u_m > 0, \quad v_m > 0;$$

3) due to the presence of the $b_2 v$ term in SC, the efficiency factors for storage and retrieval can no longer be lumped together. Let e_u and e_v be defined as the efficiency factors for storage and retrieval, respectively, with $0<e_u \leq 1$, $0<e_v \leq 1$. Hence, we have

$$g(t)+e_v v(t)=q(t)+u(t)/e_u. \tag{4'}$$

The optimality condition (10) is modified as follows:

$$\begin{aligned} H_u&=0 && \text{if } 0<u<u_m \\ H_u&\geq 0 && \text{if } u=0 \\ H_u&\leq 0 && \text{if } u=u_m. \end{aligned} \tag{10'}$$

Hence,

$$\begin{aligned} u&=0 && \text{if } H_u>0 \\ u&=u_m && \text{if } H_u<0. \end{aligned} \tag{10''}$$

Replacing u everywhere in (10′) and (10″) by v, we get (11′) and (11″). The other conditions, (12)–(14), still hold.

It can be proved that Propositions 1, 2, and 4 are still true. Proposition 3 is generalized as below. Over the storage cycle $(t_1, t_3]$ define the valley line as a straight line with slope $b_1 e_u/a_2$ and intercept

$$c_1=b_1 e_u t_1/a_2-(a_1/a_2+e_u(b_1 k+\eta)/a_2)$$

at $t=t_1$, where the constant k accounts for the jump in $\lambda(t)$. Define peak line as a straight line with slope $b_1/e_v a_2$ and intercept

$$c_2=b_1 t_1/e_v a_2-(a_1/a_2-b_2/e_v a_2+(b_1 k+\eta)/e_v a_2)$$

at $t=t_1$.

Proposition 3′: Over the storage cycle $(t_1, t_3]$ let t_2, with $t_1 \leq t_2 < t_3$, be the first instant at which the load curve crosses the valley line from below.

1) In $(t_1, t_3]$, it is optimal to store whenever the load curve falls below the valley line and to let generation follow the valley line. However, when the maximum power flow constraint (5′) is violated, store at the maximum rate $u=u_m$ and let $g(t)=q(t)+u_m/e_u$.

2) In $(t_2, t_3]$, it is optimal to retrieve whenever the load curve rises above the peak line and to let generation follow the peak line. However, when the maximum power flow constraint (6′) is violated, retrieve at the maximum rate $v=v_m$ and let $g(t)=q(t)-e_u v_m$.

3) The load curve $q(t)$ crosses the peak line at t_3.

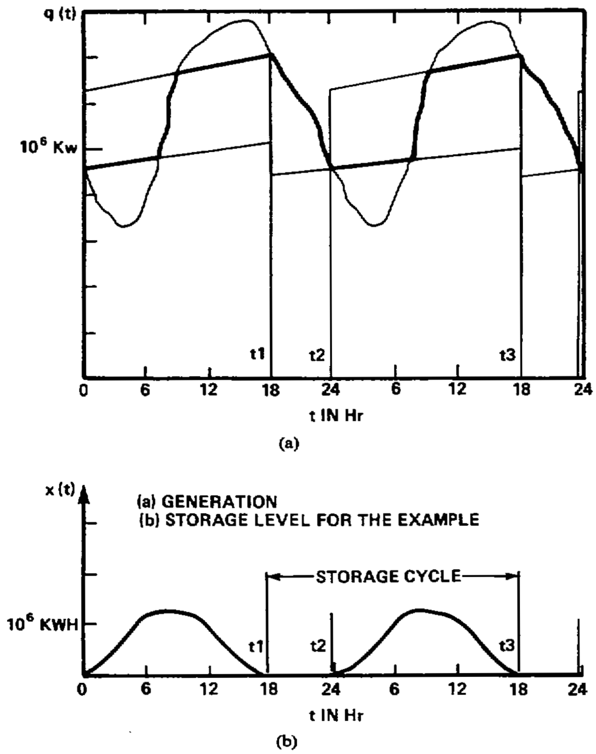

Fig. 3. (a) Generation. (b) Storage level for the example.

Note that the presence of $b_2 > 0$ does not change the slopes of the peak line and the valley line. However, due to changing intercepts, a higher b_2 means less use of the storage.

Our result can be specialized to the usual criterion of retrieving when the "system lambda"[12] exceeds some level (see, for example, [5]). In the special case $b_1=0$, the valley line and the peak line are horizontal, and Proposition 3′ can be interpreted to imply that we should retrieve whenever system lambda exceeds $a_2 c_2 + a_1$. This results in a constant system lambda until the power flow constraint u_m is binding. A similar statement can be made for the storage part of the strategy.

The optimal storage and retrieval strategies are depicted in Fig. 4 for the example in Section III-E with

$$e_u=0.9$$

$$e_v=0.8$$

$$u_m=0.16\times 10^6 \text{ (kW)}$$

$$v_m=0.25\times 10^6 \text{ (kW)}$$

$$b_2=0.$$

V. Conclusion

We have investigated the question of how storage may be used as a load management tool to shave off system peaks and fill system valleys. A dynamic formulation of the peak load problem with storage results in a standard optimal control problem with state and control variable inequality constraints. Using this optimal control problem, we have shown how peak lines and valley lines may be determined resulting in the reshaping of the total demand (customer loads and utility storage) to be met by the generation facilities.

Our model can be further generalized to include the case where the load curve $q(t)$ is generated implicitly by some "inverse demand func-

[12]System lambda is the marginal cost of generation ($/kWh) as is well known in the power literature.

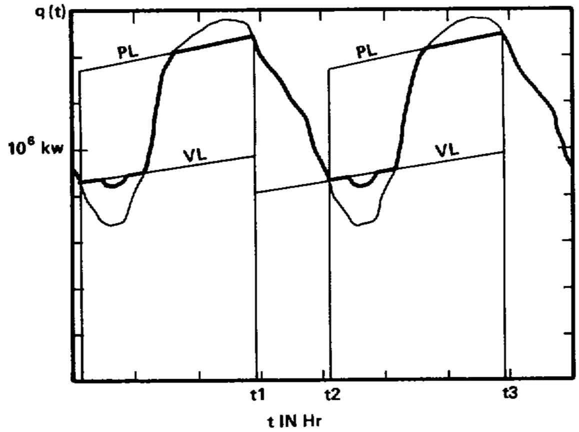

Fig. 4. Optimal generation with maximum power flow constraints.

tion" and the objective function is a "welfare function" (the sum of consumer's and producer's surpluses) [12]. It can be shown that in this case the optimality conditions still hold. The well-known marginal cost pricing rule is obtained as an additional condition which can be used to determine $q(t)$ explicitly.

It appears that our work (modified to include other further complications of the "real world" such as disaggregated generation and storage facilities, randomness in $q(t)$, etc.) may be used in two ways.

1) To guide the development of an optimal dispatch strategy for using existing storage and generation facilities.

2) To use as a planning tool for determining the best mix of generation and storage facilities.

Appendix

Proof of Proposition 3

1) First, we want to show that $u(t)>0$ if $q(t)<c_1+b_1(t-t_1)/a_2$. By way of contradiction let $u(t)=0$ for some t such that $q(t)<c_1+b_1(t-t_1)/a_2$. Then

$$H_u=a_2[q(t)-c_1-b_1(t-t_1)/a_2]-a_2ev(t)<0.$$

Thus, we have a contradiction. Now suppose $u(t)>0$, then, $H_u=0$, hence,

$$g(t)=q(t)+u(t)=c_1+b_1(t-t_1)/a_2.$$

2) The proof is analogous to 1).

3) We shall prove that $q(t)$ must cross the peak line at t_3 from above. By way of contradiction assume that $q(t)$ does not cross the peak line at t_3. Then there exists two sufficiently small positive numbers ϵ and τ such that $q(t)-g(t)>\epsilon$ for all t in $[t_3-\tau,t_3+\tau]$.

To establish a contradiction and thus complete the proof we shall construct a strategy which has less total cost than the "optimal" one. Consider a new strategy consisting of storage/retrieval functions that coincide with the optimal ones except for t in $[t_3-\tau,t_3+\tau]$, as shown in Fig. 5. It can be shown that

$$\delta J=\left[a_2q\left(t_3+\frac{\tau}{2}\right)-a_2\left[c_2+b_1\left(t_3-\frac{\tau}{2}-t_1\right)/ea_2\right]\right]\Delta\tau$$
$$-b_1(\Delta\tau)\tau/e+\text{higher order terms in }\tau$$

where $\delta J=J(\text{optimal})-J(\text{new})$. For sufficiently small τ, $\delta J>0$ and we have desired contradiction. Note that the first and second terms in the brackets are the marginal generation costs, whereas the last term is the added storage cost for the new strategy.

Proof of Proposition 4

Using (4) and Proposition 3 this proposition obviously holds within each storage cycle. What we want to show now is that it also holds at entry points.

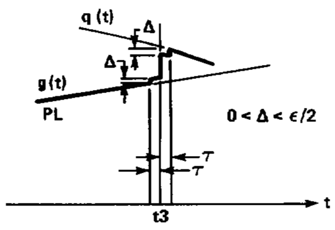

Fig. 5. Illustrating Proposition 3.

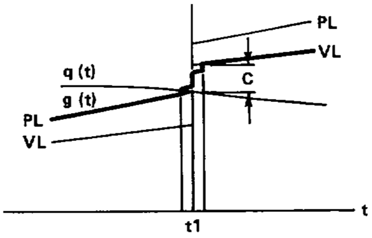

Fig. 6. Illustrating Proposition 4.

Let t_1 be the entry point under consideration. Assume $q(t_1)$ is continuous. Let t_2 be the associated exit point. There are two cases: $t_1<t_2$ and $t_1=t_2$. Note that $t_1=t_2$ is the case where the SVIC is binding on the optimal x trajectory at the single point t_1.

Case 1:

$$t_1<t_2.$$

$u(t_1)$ is continuous since $u(t_{1^-})=u(t_{1^+})=0$. $v(t_{1^+})=0$ since $v(t)=0$ for t in $(t_1,t_2]$. Also, $q(t_1)$ is continuous $v(t_{1^-})=0$ from the last part of Proposition 3. Thus $v(t_{1^-})=v(t_{1^+})=0$ and $v(t_1)$ is continuous.

Case 2:

$$t_1=t_2.$$

$v(t_1)$ is continuous as proved in the previous case. We also know $u(t_{1^-})=0$. What remains to be done is to show $u(t_{1^+})=0$. By way of contradiction assume that we have $u(t_{1^+})=c>0$. The proof, similar to that of the last part of Proposition 3, is by constructing another strategy with lower cost. For the new $g(t)$ shown in Fig. 6 it can be shown that J (optimal) $-J$ (new) >0. The proof is thus complete.

References

[1] E. E. Bailey, "Peak-load pricing under regulatory constraint," *J. Political Economy*, vol. 80, pp. 662–679, July/Aug. 1972.

[2] J. C. Panzar, "A neoclassical approach to peak load pricing," *Bell J. Economics*, vol. 7, pp. 521–530, Autumn 1976.

[3] H. Mohring, "The peak load problem with increasing returns and pricing constraints," *Amer. Economic Rev.*, vol. 60, pp. 693–705, Sept. 1970.

[4] G. D. Friedlander, "Pumped storage—An answer to peaking power," *IEEE Spectrum*, vol. 1, pp. 58–75, Oct. 1964.

[5] G. H. McDaniel and A. F. Gabrielle, "Dispatching pumped storage hydro," *IEEE Trans. Power App. Syst.*, vol. PAS-85, pp. 465–470, May 1966.

[6] M. J. Cobian, "Optimal pumped storage operation with interconnected power systems," *IEEE Trans. Power App. Syst.*, vol. PAS-90, pp. 1391–1399, May/June 1971.

[7] G. M. Morgan and S. N. Talukdar, "Electric power load management: Some technical, economic, regulatory and social issues," *Proc. IEEE*, vol. 67, pp. 241–312, Feb. 1979.

[8] J. F. Dopazo, S. T. Ehrmann, and A. M. Sasson, "Real-time power systems generation scheduling optimization," in *Proc. 7th Annu. Pittsburgh Conf. Modeling Simulation*, vol. 7, pt. 1, 1976, pp. 594–598.

[9] B. M. Mitchell, W. G. Manning, Jr., and J. P. Acton, *Peak-Load Pricing. European Lessons for U.S. Energy Policy* (A Rand Corporation research study). Cambridge, MA: Ballinger.

[10] A. E. Bryson, Jr. and Y.-C. Ho, *Applied Optimal Control.* New York: Wiley, 1975 (a Halsted Press book).

[11] D. G. Luenberger, *Optimization by Vector Space Methods.* New York: Wiley, 1969.

[12] I. Pressman, "A mathematical formulation of the peak-load pricing problem," *Bell J. Economics Management Sci.*, vol. 1, pp. 304–326, Autumn 1970.