$$S(N)=Q_1, \quad t=N-1,\cdots,t_0 \tag{11}$$

and W and Z are $p\times l$ and $p\times(p-l)$ matrices, respectively, such that $[W \vdots Z]$ is nonsingular, and $H^TBW=I$ and $H^TBZ=0$. The minimum values of the objective functions are

$$\min_{U\in\mathscr{U}} J_i = \sum_{t=t_0}^{N} \operatorname{tr} H_iH_i^TP(t) + \sum_{t=t_0}^{N-1} \operatorname{tr} H_iH_i^TK(t) \cdot P_y(t+1-L)K(t)^T + m^TH_iH_i^Tm, \quad i=1,\cdots,k \tag{12}$$

$$\min_{U\in\mathscr{U}'} J_{k+1} = \sum_{t=t_0}^{N} \operatorname{tr} Q_1P(t) + \sum_{t=t_0}^{N-1} \operatorname{tr} S(t+1)K(t) \cdot P_y(t+1-L)K(t)^T + m^TS(t_0)m \tag{13}$$

where $P(t)$ is the covariance of the estimation error $x(t)-\hat{x}(t)$, $P_y(t+1-L)$ is the covariance of the innovations $\tilde{y}(t+1-L)=y(t+1-L)-E[y(t+1-L)|\mathscr{P}_t]$, $K(t)$ is the Kalman filter gain, and $m=Ex(t_0)$.

Proof: Due to (1) and (3) the set $\mathscr{U}'$ defined by (6) is

$$\mathscr{U}'=\{U|H^T(A\hat{x}(t)+Bu(t))=0, \quad t=N-1,\cdots,t_0\} \tag{14}$$

and the minimum values of J_i, $i=1,\cdots,k$ are given by (12) [3]. The assumptions on the matrices H and B are such that $\mathscr{U}'$ is nonempty. The minimum of J_{k+1} in the set $\mathscr{U}'$ is given by dynamic programming [3]. Due to (14) the solution of the Bellman equation is obtained recursively by solving a quadratic programming problem [4] at each step, giving the result of the theorem. □

Note that the condition on rank H^TB can be relaxed, and it is only necessary to assume that the l ($\leqslant p$) states H^Tx are controllable from all inputs. The general case can be solved as above, but the set $\mathscr{U}'$ is modified and may be time varying.

III. Example

The following simple example illustrates the procedure. Consider a system described by (1) with

$$A=\begin{bmatrix}0.9 & 0.2\\ -0.6 & 0.8\end{bmatrix}, \quad B=\begin{bmatrix}1 & -0.5\\ 2 & 0.9\end{bmatrix},$$

$$C=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad R_w=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \quad R_v=0$$

and $\mathscr{P}_t=\{y(t),\cdots\}$, i.e., complete state information. Consider the loss functions

$$J_1=\lim_{N\to\infty} E\left[\frac{1}{N}\sum_{t=1}^{N} x_1(t)^2\right]$$

$$J_2=\lim_{N\to\infty} E\left[\frac{1}{N}\sum_{t=1}^{N}\left[q_1x_2(t)^2+u(t)^TQ_2u(t)\right]\right].$$

Now $k=1$, $l=1$, $H=[1 \quad 0]^T$ and one specific choice for the matrices W and Z is

$$W=B^TH(H^TBB^TH)^{-1}=[0.8 \quad -0.4]^T$$

$$Z=[1 \quad 2]^T.$$

Taking J_1 as the primary objective function, (14) gives the set $\mathscr{U}'$ of control strategies for which the variance of the state x_1 attains its achievable minimum value

$$J_1(\mathscr{U}')=\min_{U\in\mathscr{U}} J_1=1.$$

The design weights q_1 and Q_2 of the secondary loss function J_2 can then be selected in the usual way [1], such that the variances of x_2, u_1, and u_2 are jointly acceptable. For example, for $q_1=1$, $Q_2=I$ the optimal strategy and the minimum value of J_2 are

$$u^*(t)=-\begin{bmatrix}0.376 & 0.271\\ -1.05 & 0.142\end{bmatrix}x(t)$$

$$\min_{U\in\mathscr{U}'} J_2=2.54.$$

IV. Conclusions

The solution to a multiobjective optimal control problem with a hierarchical ordering of the individual loss functions has been given. A potential field of application of the result is in industrial process control, where it is well known that substantial gains may result from a reduction of the variances of some variables [3]. It could then be meaningful to take these variances as the primary objective functions J_i, $i=1,\cdots,k$, and the other variables could be included in the secondary objective function J_{k+1}.

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Three-Level Stackelberg Decision Problems

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Abstract—**Three-level Stackelberg decision problems are studied by using the inducible region concept. Through a systematic derivation, it is identified that the leader's control has dual purposes, which in general are not separable. A special class of problems is then considered, where explicit results are obtained.**

I. Introduction

Recently, most results on Stackelberg decision problems were obtained by using the "team solution approach." The essential notion is to have the leader modify the followers' cost functions (through appropriate leader's strategy) so that the followers are induced to behave as if they were also optimizing the leader's cost function. Sufficient conditions for achieving the team solution have then been derived under several formulations ([1], [5], [8], [11] for two-level problems and [2] for three-level problems).

To handle the situation where the team solution approach fails, another methodology began to emerge [9], [10]. The concept of *inducible region* was then formally introduced in [3], [4].[1] By using this concept, we have

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[1]For an historic discussion on the development, please see [4, sect. VI].

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obtained *necessary* and *sufficient* conditions for the existence of a Stackelberg solution for two-person single-stage as well as multistage problems [4], [6]. In this note, we investigate three-level Stackelberg problems. The problem formulation is given in Section II. The inducibility conditions are derived in Section III. These conditions give a clear picture of the problem and indicate its fundamental difficulties. In particular, we identify that the leader's control has *dual purposes*: direct control and indirect influence. The fact that the dual purposes are generally not separable makes the problem difficult to solve. We then in Section IV obtain explicitly solutions for a special class of problems where the dual controls degenerate. Concluding remarks are given in Section V.

II. Problem Formulation

Consider a three-person decision problem with three levels of hierarchy. The leader ($DM0$) is at the top of the hierarchy; the first follower ($DM1$) and the second follower ($DM2$) are, respectively, at the middle level and the bottom of it. Let the strategy of DMi be denoted as r_i, with $r_i \in \Gamma_i$; the decision of DMi as u_i, with $u_i \in U_i$; and the cost function as $J_i(r_0, r_1, r_2)$ in its strategy form or $J_i(u_0, u_1, u_2)$ in its extensive form. The solution concept adopted here is the same as that of [2] or the first part of [10]. As mentioned earlier, the team solution approach was used in [2]. On the other hand, as we go through the derivation here, it will become clear that [9, Theorem 3] is incorrect.

Consider a complete information problem where each decision maker acts only once.[2] The sequence of actions is shown in Fig. 1. In the design stage, $DM0$ designs and announces r_0 first. Knowing r_0, $DM1$ designs and announces r_1. In the execution stage, $DM2$ acts first, knowing both r_0 and r_1. $DM1$ then calculates u_1 according to his announced r_1, i.e., $u_1 = r_1(u_2)$. Finally, $DM0$ calculates u_0 according to $u_0 = r_0(u_1, u_2)$.[3] The strategy r_i is allowed to be any measurable function from DMi's information set to U_i (mixed strategies will not be considered here).

As discussed in [3], [4], the inducible region is the collection of all the possible outcomes. In this case, for a given r_0, since the reactions of $DM1$ and $DM2$ might not be unique, the realized outcome might thus not be unique. According to the solution concept (see [2, eqs. (1) and (2)]), however, $DM0$ *assumes* that followers choose a particular pair (r_1', r_2') in his strategy design stage, where[4]

$$(r_1', r_2') = \underset{r_1 \in R_1(r_0)}{\arg\sup} \; \sup_{r_2 \in R_2(r_0, r_1)} J_0(r_0, r_1, r_2). \tag{2.1}$$

In the above equation, $R_1(r_0)$ and $R_2(r_0, r_2)$ represent, respectively, $DM1$'s and $DM2$'s reaction sets, and are defined by [2, eq. 2]. The corresponding decision tuple (u_0', u_1', u_2') is thus called *the outcome from the leader's viewpoint* for this r_0. As in [4], we can formulate equivalent classes and, with a little abuse of notation, say that from the leader's viewpoint, the *unique* outcome (u_0', u_1', u_2') is *induced* by this r_0. The inducible region IR is then defined as

$$\mathrm{IR} = \{(u_0, u_1, u_2)\text{: There exists a } r_0 \text{ such that } (u_0, u_1, u_2) \text{ is the unique outcome induced by this } r_0\}. \tag{2.2}$$

III. Inducibility Conditions

For any given r_0, $DM1$ and $DM2$ face a two-level Stackelberg problem (the "nested" two-level problem). $DM0$ can then be conceived of as also facing a two-level problem (the outer layer problem), where the reactions of the followers [in terms of (r_1, r_2) or (u_1, u_2)] are determined by the nested two-level problem. Consequently, the three-level problem can be analyzed by considering these two two-level problems.

We shall first consider the nested two-level problem. This problem has been solved completely in [4]. To convey main ideas, we shall assume for simplicity that all the parameter optimization subproblems treated

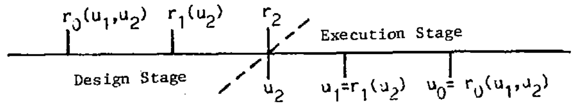

Fig. 1.

hereafter have solutions.[5] With this assumption, the inducible region for $DM1$ for any fixed r_0 can be written as

$$\mathrm{IR}_1(r_0) = \{(u_1, u_2) \colon u_i \in U_i, i = 1,2 \text{ and } J_2(r_0(u_1, u_2), u_1, u_2) < m_2(r_0)\} \; U \, S_2^*(r_0), \tag{3.1}$$

where

$$m_2(r_0) \triangleq \min_{u_2} \max_{u_1} J_2(r_0(u_1, u_2), u_1, u_2)^6 \tag{3.2}$$

and $S_2^*(r_0)$, which is basically the solution set of (3.2) that maximizes J_1, is defined rigorously in [4].

Intuitively, no matter what $DM1$ is going to do, $DM2$ can always guarantee himself that J_2 will not be greater than $m_2(r_0)$. Thus, any point with $J_2 > m_2(r_0)$ is not inducible. On the other hand, any point with $J_2 < m_2(r_0)$ is inducible since $DM1$ can always penalize $DM2$ to $m_2(r_0)$ if $DM2$ does not select the designated point. On the $J_2 = m_2(r_0)$ boundary, only points in $S_2^*(r_0)$ are inducible. For proof and detailed discussions, see [4].

Let us now turn to the outer layer problem, and examine whether a point $u' = (u_0', u_1', u_2')$ is inducible or not. The candidate r_0' that induces (u_0', u_1', u_2') has to take the following form:

$$r_0'(u_1, u_2) = \begin{cases} u_0', & \text{if } (u_1, u_2) = (u_1', u_2'), \\ P_0(u_1, u_2) & \text{otherwise,} \end{cases} \tag{3.3}$$

where the punishing strategy P_0 is yet to be specified. In order to induce u', the following two conditions have to be satisfied.

C1) Capability Condition: $DM1$ must be able to induce (u_1', u_2'), i.e.,

$$(u_1', u_2') \in \mathrm{IR}_1(r_0').$$

C2) Desirability Condition: $DM1$ must prefer (u_1', u_2') to any other points in $\mathrm{IR}_1(r_0')$, i.e.,

$$J_1(u') \leq J_1(P_0(u_1, u_2), u_1, u_2) \qquad \forall (u_1, u_2) \in \mathrm{IR}_1(r_0'), \quad (u_1, u_2) \neq (u_1', u_2').$$

We can now derive some explicit results. First C1 and (3.1) imply that

$$J_2(u') \leq \min_{u_2} \max_{u_1} J_2(r_0'(u_1, u_2), u_1, u_2) \tag{3.4}$$

$$\leq \min_{u_2} \max_{u_1} \max_{u_0} J_2(u_0, u_1, u_2) \triangleq M_2. \tag{3.5}$$

The constant M_2 can be thought of as $DM2$'s guaranteed cost, since $DM2$ can always get M_2 even if both $DM0$ and $DM1$ maximize J_2. Therefore, it is impossible for $DM0$ to induce any point with $J_2 > M_2$. We thus have the following:

$$\mathrm{IR} \subset \{u \colon J_2(u) \leq M_2\}. \tag{3.6}$$

The condition (C2) implies that

$$J_1(u') \leq \min_{\substack{(u_1, u_2) \in \mathrm{IR}_1(r_0') \\ (u_1, u_2) \neq (u_1', u_2')}} J_1(P_0(u_1, u_2), u_1, u_2). \tag{3.7}$$

[2] It can be extended to problems with incomplete information, see [7].

[3] We assume that both $DM0$ and $DM1$ keep their promises by carrying out their announced strategies.

[4] For simplicity, we assume that such a (r_1', r_2') exists.

[5] Without this assumption, the problem can still be handled by following the method of [4], however, the treatment of the inducible region's boundary becomes very complicated.

[6] If there are multiple u_1's that maximize J_2, the one that minimizes J_1 will be selected.

That is, the punishing strategy P_0 should make all other alternatives in $\mathrm{IR}_1(r_0')$ less attractive to $DM1$ than the desired u'. Note that P_0 affects the term on the right-hand side of (3.7) *directly* since it determines J_1's first argument. P_0 also affects the term *indirectly* since it shapes $\mathrm{IR}_1(r_0')$ and thus changes the region of taking the minimum. Intuitively, if $DM2$'s cost function depends on $DM0$'s control variables, then $DM0$ can modify $DM2$'s cost through r_0, and this in turn affects $DM1$'s ability to induce $DM2$'s decisions. Thus, $DM0$ has *indirect influence on DM1's decisions*. On the other hand, if $DM1$'s cost depends on $DM0$'s control variables, $DM0$ can assign adequate values to them (through r_0) as threats, and thus imposes *direct control on DM1's decisions* (in contrast to the indirect influence). In order to select a good punishing strategy P_0, we therefore have to consider both effects, and *the dual purposes of DM0's strategy* are observed.

In order to delineate the entire inducible region, we have to find out that for a given u', what is the best the leader can do to make all other alternatives less attractive. The *most* $DM0$ can do is to impose upon $DM1$ the punishing strategy that maximizes the right-hand side of (3.7), i.e.,

$$\max_{P_0}\left\{\min_{\substack{(u_1,u_2)\in \mathrm{IR}(r_0')\\ (u_1,u_2)\neq(u_1',u_2')}} J_1\big(P_0(u_1,u_2),u_1,u_2\big)\right\}. \tag{3.8}$$

The indirect influence changes $\mathrm{IR}_1(r_0')$. To the best of the authors' knowledge, there does not exist any systematic way in selecting the "best" P_0 for shaping the optimization region. In addition, the generally nonseparable dual roles of $DM0$'s strategy further complicates the issue.

IV. Inducible Region for a Class of Problems

Consider the problem in which $DM0$ cannot affect $DM2$ directly, i.e., the case where $J_2 = J_2(u_1, u_2)$. From (3.1) and (3.2), we see that now IR_1 is independent of r_0. Consequently, (3.8) becomes

$$\max_{P_0}\left\{\min_{\substack{\mathrm{IR}_1\\ (u_1,u_2)\neq(u_1',u_2')}} J_1\big(P_0(u_1,u_2),u_1,u_2\big)\right\}, \tag{4.1}$$

and the dual control degenerates to direct control. We can rewrite this optimization problem into its extensive form:

$$\min_{\substack{(u_1,u_2)\in \mathrm{IR}_1\\ (u_1,u_2)\neq(u_1',u_2')}} \max_{u_0} J_1(u_0,u_1,u_2). \tag{4.2}$$

From (3.7) and (4.2), we then conclude that

$$J_1(u') \leqslant \min_{(u_1,u_2)\in \mathrm{IR}_1} \max_{u_0} J_1(u_0,u_1,u_2) \triangleq M_1. \tag{4.3}$$

M_1 can be thought of as $DM1$'s guaranteed cost. Let $u_{0P}(u_1, u_2)$ denote the solution to the maximization subproblem in (4.3).[7] Also let S_1^* denote the solution to the minimax problem in (4.3) that maximizes J_0. Then $DM0$'s maximum penalty strategy is given by

$$P_0^*(u_1,u_2)=\begin{cases} u_{0P}(u_1,u_2), & \text{if } (u_1,u_2)\in \mathrm{IR}_1,\\ \text{arbitrary} & \text{otherwise.}\end{cases} \tag{4.4}$$

We then have the following results.

Theorem 1: If J_2 is not a function of u_0, then

$$\mathrm{IR} = \{(u_0,u_1,u_2)\colon u_i\in U_i,\ \ i=1,2,3,\ J_1(u_0,u_1,u_2)\leqslant M_1 \text{ and } J_2(u_1,u_2)\leqslant M_2. \text{ The first equality holds only for } (u_0,u_1,u_2)\in S_1^*, \text{ and the second equality holds only for } (u_1,u_2)\in S_2^*\}. \tag{4.5}$$

The proof is given in [7]. Consequently, one way to solve such a Stackelberg problem is to go through the following steps.

S1): Delineate IR as in (4.5).

[7] If there are multiple u_0's that maximize J_1, the one that minimizes J_0 will be selected.

S2): Find the optimal inducible outcome u^* by solving the following parameter optimization problem:

$$u^* = \arg \min_{(u_0,u_1,u_2)\in \mathrm{IR}} J_0(u_0,u_1,u_2). \tag{4.6}$$

S3): Construct a r_0^* to induce u^*. One such r_0^* is given by

$$r_0^*(u_1,u_2)=\begin{cases} u_0^* & \text{if } (u_1,u_2)=(u_1^*,u_2^*),\\ P_0^*(u_1,u_2) & \text{otherwise.}\end{cases} \tag{4.7}$$

It can be readily seen that a Stackelberg strategy r_0^* exists *if and only if* (4.6) has a solution in IR. Furthermore, if the Stackelberg cost J_0^* is bounded (J_0^* as defined by [2, eq. (1)]), then an ϵ-Stackelberg strategy *always* exists.

V. Concluding Remarks

Through a systematic derivation, we reveal the nature of three-level problems, and indicate its fundamental difficulties. In particular, the dual purposes of the leader's strategy, *direct control* and *indirect influence*, are identified. Together with other results obtained by using the inducible region concept, it is believed that this concept is fundamental to the study of many Stackelberg decision problems.

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Digitalization of Existing Continuous Control Systems

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Abstract—**The problem of converting existing continuous-data control systems into digital control systems is considered. The objective of this paper is to develop a computer-aided method for synthesizing the pulse-transfer function of the digital controller. This is done by matching the frequency response of the digital control system to that of the continuous system with a minimum weighted mean-square error. Formulas for computing the parameters of the digital controller are obtained as a result. The design technique is illustrated with a numerical example, and a comparison with previous methods is also presented.**

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