$$s_2(0)=-0.25F^2\left[-\left(A_2+\sqrt{A_2^2+4\rho_2}\right)t_f -\ln\frac{\left(1-\alpha_2\frac{A_2-\sqrt{A_2^2+4\rho_2}}{2}\right)+\left(\alpha_2\frac{A_2+\sqrt{A_2^2+4\rho_2}}{2}-1\right)e^{-2\sqrt{A_2^2+4\rho_2}t_f}}{\sqrt{A_2^2+4\rho_2}}\right]. \tag{31}$$

It can be proved that if

$$\frac{1}{\alpha_1}>\frac{A_i^*+\sqrt{A_i^{*2}+4\rho_1}}{2}$$

and

$$\frac{1}{\alpha_2}>\frac{A_2+\sqrt{A_2^2+4\rho_2}}{2}$$

then

$$\frac{dH_i^*(t)}{dt}>0,\quad \frac{dH_2(t)}{dt}>0$$

for all $t\in[0,t_f]$. Hence,

$$\max_{t\in T}H_i^*(t)=\frac{1}{\alpha_1},\quad \max_{t\in T}H_2(t)=\frac{1}{\alpha_2},\quad \delta_i=4\frac{1}{\alpha_1},\quad \delta_2=\frac{2}{\alpha_2},\quad T=[0,t_f].$$

Noting that

$$|C_{12}|=2\quad\text{and}\quad |C_{21}|=\left(\sum_{i=1}^{4}\beta_i^2\right)^{1/2}$$

and matrix S defined by vi) has the form

$$S=\begin{bmatrix} -\rho_1, & \dfrac{4}{\alpha_1}+\dfrac{1}{\alpha_2\left(\sum_{i=1}^{4}\beta_i^2\right)^{1/2}} \\ \dfrac{4}{\alpha_1}+\dfrac{1}{\alpha_2\left(\sum_{i=1}^{4}\beta_i^2\right)^{1/2}}, & -\rho_2\end{bmatrix}, \tag{32}$$

it now follows from the theorem (and the corollary) that the initial state $[y_{10}^*, y_{20}^*, y_{30}^*, y_{40}^*, y_{50}^*]$ of the composite system (14), (15) is stochastically ϵ-controllable in probability φ in the normed squared sense with respect to the terminal state in the time interval $[0,t_f]$ if S is negative defined, i.e., if

$$\rho_1\rho_2>\left(\frac{4}{\alpha_1}+\frac{1}{\alpha_2\left(\sum_{i=1}^{4}\beta_i^2\right)^{1/2}}\right)^2 \tag{33}$$

and conditions ii) and iv) hold, i.e.,

$$\exists\varphi\quad 0<\varphi=1-\frac{\alpha_1}{\epsilon_1}\left(\sum_{i=1}^{4}y_{i0}^{*2}H_i^*(0)+s_i(0)\right)-\frac{\alpha_2}{\epsilon_2}\left(y_{20}^2H_2(0)+s_2(0)\right) \tag{34}$$

and

$$\frac{\alpha}{\epsilon}=\min\left(\frac{\alpha_1}{\epsilon_1},\frac{\alpha_2}{\epsilon_2}\right),\quad \frac{1}{\alpha}=\min\left(\frac{1}{\alpha_1},\frac{1}{\alpha_2}\right).$$

Hence,

$$\frac{1}{\epsilon}=\min\left(\frac{\alpha_1}{\epsilon_1},\frac{\alpha_2}{\epsilon_2}\right)\min\left(\frac{1}{\alpha_1},\frac{1}{\alpha_2}\right). \tag{35}$$

V. Final Remarks

The problem of ϵ-controllability for continuous parameter stochastic composite systems has been considered. The sufficient conditions of ϵ-controllability obtained in this paper are an extension of the results [10] for linear stochastic composite systems. In a similar way, using the Gershwin and Jacobson results [4], it is possible to introduce the definitions and to prove the sufficient conditions of controllability for nonlinear deterministic and stochastic systems.

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Derivation of Necessary and Sufficient Conditions for Single-Stage Stackelberg Games Via the Inducible Region Concept

TSU-SHUAN CHANG AND PETER B. LUH

Abstract—**The inducible region is defined as the collection of all the possible outcomes. It is typically a subset of the entire decision space. The best the leader can obtain is then the optimal outcome in this inducible region. Necessary and sufficient conditions are derived by first delineating the inducible region, and then obtaining a leader's optimal strategy, if it exists. If not, an ϵ-strategy always exists, provided that the leader's Stackelberg cost is bounded.**

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I. Introduction

Stackelberg games can be used to model multiperson optimization problems having decision makers in different levels of hierarchy. The central theme lies in the assumption that the leader, occupying the higher level of hierarchy, can choose his strategy to optimize his operation by taking into account the rational reactions of followers. Although the concept had been introduced earlier in [1], it was believed until recently that even a simple looking two-person, linear, quadratic, deterministic, closed-loop Stackelberg game was difficult to solve [2]. Recent progress is mainly due to the discoveries of indirect methods of solving deterministic Stackelberg games, which are introduced independently in [3], [4]. (A few results using direct methods such as [5] are also reported.) These indirect methods have been extended to solve more deterministic and stochastic Stackelberg games such as in [6]–[9]. Conceptually, previous results try to find sufficient conditions to guarantee the existence of a solution or an ϵ-solution. What is lacking in the literature are the necessary and sufficient conditions.

In this note, we demonstrate how necessary and sufficient conditions can be derived through the delineation of the *inducible region*. To achieve this, we state two-person, single-stage, deterministic games in Section II. We then define mathematically the inducible region in Section III. Under the assumption that every optimization problem considered has a unique solution, we derive in Section IV inducibility conditions and thus the inducible region step by step from the definition. In Section V, the assumption of uniqueness is dropped and we then present necessary and sufficient conditions for general problems. When a Stackelberg solution does not exist, we show that an ϵ-Stackelberg solution always exists under certain conditions. Concluding remarks in Section VI include a comparison between Tolwinski's results [7] and ours.

II. Two-Person, Single-Stage, Deterministic Games

Since a two-person, single-stage, deterministic Stackelberg game with incomplete information can be converted into an equivalent one with complete information by following [6], it suffices here to consider the problem with complete information. Let $J_0(u_0, u_1)$ and $J_1(u_0, u_1)$ be the cost functions of the leader and the follower, respectively. For any given leader's strategy $\gamma_0 \in \Gamma_0$, the follower chooses his control variable $u_1 \in U_1$ to minimize J_1. The leader, who is assumed to know the follower's control u_1 precisely, then decides on the value of his control $u_0 \in U_0$ according to his preannounced strategy, i.e., $u_0 = \gamma_0(u_1)$. Here we assume that γ_0 is the set of all functions from U_1 to U_0. The problem for the leader is to find his Stackelberg strategy γ_0^s to satisfy

$$\sup_{u_1 \in R(\gamma_0^s)} J_0(\gamma_0^s(u_1), u_1) \leqslant \sup_{u_1 \in R(\gamma_0)} J_0(\gamma_0(u_1), u_1) \qquad \text{for all } \gamma_0 \in \Gamma_0, \tag{1}$$

where

$$R(\gamma_0) = \left\{ u_1' \in U_1 ; J_1\left(\gamma_0\left(u_1'\right), u_1'\right) \leqslant J_1\left(\gamma_0\left(u_1\right), u_1\right), \quad \text{for all } u_1 \in U_1 \right\} \tag{2}$$

is the rational reaction set of the follower for the given γ_0. Any $u_1^s \in R(\gamma_0^s)$ is called a Stackelberg solution of the follower, and the Stackelberg cost of the leader is defined as

$$J_0^s = \inf_{\gamma_0} \sup_{u_1 \in R(\gamma_0)} J_0(\gamma_0(u_1), u_1). \tag{3}$$

In the case where a Stackelberg strategy does not exist, we consider the ϵ-Stackelberg strategy γ_0^ϵ which satisfies

$$\sup_{u_1 \in R(\gamma_0^\epsilon)} J_0(\gamma_0^\epsilon(u_1), u_1) < J_0^s + \epsilon, \qquad \text{provided that } |J_0^s| < \infty. \tag{4}$$

III. The Concept of Inducible Region

For a given γ_0, the rational follower can choose any $u_1 \in R(\gamma_0)$. If the follower chooses a particular $u_1' \in R(\gamma_0)$, then the leader has to act according to $u_0' = \gamma_0(u_1')$ (under the assumption that he keeps his commitment). In other words, (u_0', u_1') is the *real outcome* of the game for this γ_0. However, in the strategy designing stage, the leader's viewpoint regarding the outcome of the game is somewhat different. From (2), the leader assumes that the follower chooses a particular $u_1^* \in G(\gamma_0)$, where

$$G(\gamma_0) \triangleq \underset{u_1 \in R(\gamma_0)}{\arg\max}\, J_0(\gamma_0(u_1), u_1) \tag{5}$$

is assumed to be nonempty for the convenience of discussion. Since $J_0(\gamma_0(u_1), u_1)$ as well as $J_1(\gamma_0(u_1), u_1)$ takes the same value for all $u_1^* \in G(\gamma_0)$, all the outcomes in the set $\{(u_0^*, u_1^*) | u_0^* = \gamma_0(u_1^*), u_1^* \in G(\gamma_0)\}$ are equivalent. With little abuse of notations, we shall use (u_0^*, u_1^*) to represent the equivalent set, and say that, *from the leader's viewpoint*, the *unique outcome* (u_0^*, u_1^*) can be *induced* by announcing the given γ_0. Let us define the *inducible region* (IR) by[1]

$$\text{IR} = \{(u_0^*, u_1^*) | \text{ there exists a } \gamma_0 \in \Gamma_0 \text{ such that } (u_0^*, u_1^*) \text{ is the unique outcome from the leader's viewpoint}\}. \tag{6}$$

From the above definition, any point not belonging to IR cannot be the outcome of the game (from the leader's viewpoint). Thus, the leader needs only to consider IR when he tries to minimize J_0, i.e.,

$$J_0^s = \inf_{(u_0, u_1) \in \text{IR}} J_0(u_0, u_1). \tag{7}$$

Therefore, the problem boils down to how to delineate the inducible region IR.

IV. Delineation of the Inducible Region

To get across main ideas, we shall assume in this section that every optimization problem considered here has a unique solution. This assumption will be dropped in the next section. First we derive the conditions on the inducibility of a given point (u_0^*, u_1^*).

The candidate γ_0^* used to induce (u_0^*, u_1^*) has to take the form

$$\gamma_0^*(u_1) = \begin{cases} u_0^* & \text{if } u_1 = u_1^*, \\ P(u_1) & \text{otherwise}, \end{cases} \tag{8}$$

where $P(u_1)$ is a function from U_1 to U_0 (yet to be specified). For any given $P(\cdot)$, the following three exhaustive cases can happen for each $u_1' \neq u_1^*$.

Case 1: $J_1(u_0^*, u_1^*) < J_1(u_0', u_1')$, where $u_0' \triangleq P(u_1')$.

Case 2: $J_1(u_0^*, u_1^*) = J_1(u_0', u_1')$.

Case 3: $J_1(u_0^*, u_1^*) > J_1(u_0', u_1')$.

If Case 3 is true for at least *one* $u_1' \neq u_1^*$, then (u_0^*, u_1^*) cannot be induced by this γ_0^* (because the follower will then prefer u_1' to u_1^*). Thus, the leader will try his best in the selection of $P(\cdot)$ to avoid Case 3 from happening. For each $u_1' \neq u_1^*$, the highest value that the leader can assign to J_1 is

$$m(u_1') = \sup_{u_0} J_1(u_0, u_1'). \tag{*}$$

In other words, the most the leader can choose for each u_1' is u_{0p}, where

$$u_{0p}(u_1') = \arg \sup_{u_0} J_1(u_0, u_1'). \tag{9}$$

Thus, the best γ_0 the leader can choose (to avoid Case 3 from happening) is

$$\gamma_{0M}(u_1) = \begin{cases} u_0^* & \text{if } u_1 = u_1^*, \\ u_{0p}(u_1) & \text{otherwise}. \end{cases} \tag{10}$$

From the construction of γ_{0M}, *if (u_0^*, u_1^*) cannot be induced by γ_{0M}, then it cannot be induced by any other γ_0, and thus it is not inducible.* This leads to the two results given as follows.

[1] The concept of inducible region can be elaborated in both strategy space and decision space. Please see [12] for details. Also, once the "unique" outcome (u_0^*, u_1^*) is defined in the equivalent sense as described, the inducible region defined by (6) can be interpreted as the collection of all the "incentive controllable" points (incentive controllability as defined in [8]).

a) If $J_1(u_0^*, u_1^*) > M$, where

$$M = \inf_{u_1} m(u_1) \tag{**a}$$

$$= \inf_{u_1} \sup_{u_0} J_1(u_0, u_1), \tag{**b}$$

then (u_0^*, u_1^*) is not inducible (due to the unavoidable occurrence of Case 3).

b) If $J_1(u_0^*, u_1^*) < M$, then (u_0^*, u_1^*) is inducible (by γ_{0M} since Case 1 holds for all $u_1' \neq u_1^*$).

Results a) and b) imply that the remaining part of IR that needs to be delineated is the boundary where $J_1(u_0^*, u_1^*) = M$, i.e., when Case 2 occurs. From the construction of γ_{0M}, if u_{1M} is the unique solution which achieves the minimum of (**a), then $(\gamma_{0M}(u_{1M}), u_{1M})$ is the minimax solution of (**b). Thus, for a point (u_0^*, u_1^*) on the boundary, we have $J_1(u_0^*, u_1^*) = J_1(\gamma_{0M}(u_{1M}), u_{1M}) = M$. According to (5), we can conclude the following.

c) A point (u_0^*, u_1^*) is inducible (as the unique outcome from the leader's viewpoint) by γ_{0M} iff $J_0(u_0^*, u_1^*) > J_0(\gamma_{0M}(u_{1M}), u_{1M})$.

From previous discussions, we can conclude that, under the assumption that the optimization problem considered has a unique solution, the IR is completely delineated by a)–c).

V. Necessary and Sufficient Conditions

From the construction procedure of γ_{0M} in (10), we can see that the assumption that (*) has the unique solution is important. For a general problem, (*) may have either multiple solutions or no solution at all. If (*) has no solution, then (9) is not well defined. If (*) has multiple solutions, we have to determine which one of them will be used to construct γ_{0M}. Therefore, it is necessary to distinguish between these two cases.

Let $\hat{U}_1$ denote the set of all u_1's for which $m(u_1')$ in (*) *cannot* be achieved by any $u_0 \in U_0$; furthermore assume that $m(u_1') < \infty$ on U_1. By the definition of supremum, there *always* exists a $u_{0\epsilon}(u_1') \in U_0$ which achieves $m(u_1')$ within ϵ in the following sense:

$$|J_1(u_{0\epsilon}, u_1') - m(u_1')| < \epsilon. \tag{11}$$

Apparently, this $u_{0\epsilon}$ is the leader's best choice to construct γ_{0M}.

Let $U_{1a} = U_1 - \hat{U}_1$, then U_{1a} is the set where $m(u_1')$ in (*) can be achieved by at least one u_0 from the definition of $\hat{U}_1$. For each $u_1' \in U_{1a}$, the leader will choose a $u_{0M}(u_1')$ to construct γ_{0M}, where

$$u_{0M}(u_1') = \arg \inf_{u_0'} J_0(u_0', u_1'), \tag{12a}$$

$$\text{subject to } u_0' \in \arg\max_{u_0} J_1(u_0, u_1'). \tag{12b}$$

Since all u_0' in (12b) achieve $m(u_1')$, the leader will certainly choose the one in (12a) to minimize J_0. Note that if the solution in (12a) is not unique, $u_{0M}(u_1')$ can be any one of them. If the solution of (12a) does not exist, then $u_{0M}(u_1')$ is chosen to make $J_0(u_{0M}(u_1'), u_1')$ achieve the infimum in (12a) within ϵ. From (11) and (12), we then obtain the modified γ_{0M} as follows:

$$\gamma_{0M}(u_1) = \begin{cases} u_0^* & \text{if } u_1 = u_1^* \\ u_{0M}(u_1) & \text{if } u_1 \neq u_1^*, u_1 \in U_{1a}, \\ u_{0\epsilon}(u_1) & \text{if } u_1 \neq u_1^* u_1 \in \hat{U}_1. \end{cases} \tag{13}$$

It can be seen that a) and b) of Section IV still hold with this new γ_{0M}.

If (u_0^*, u_1^*) is on the boundary with $J_1(u_0^*, u_1^*) = M$, then we have to modify the inducibility conditions c) in Section IV according to the existence of the solution of (**a). Denote the u_1's which achieve the infimum of (**a) by u_{1M}, then we have the following three exhaustive and exclusive cases:

Case A: there is a $u_{1M} \in \hat{U}_1$,

Case B: there is no $u_{1M} \in \hat{U}_1$, but there is a $u_{1M} \in U_{1a}$,

Case C: there is no $u_{1M} \in U_1$.

If Case A is true, then there is no u_0 which makes $J_1(u_0, u_{1M})$ equal to M according to the definition of $\hat{U}_1$. Therefore, $J_1(u_0, u_{1M}) < M$ for all $u_0 \in U_0$. This implies that (u_0^*, u_1^*) is not inducible because the follower prefers u_{1M} to u_1^*. In other words, we have Lemma 1.

Lemma 1: If Case A is true, then IR $= D_0$, where

$$D_0 \triangleq \{(u_0, u_1) | J_1(u_0, u_1) < M\}. \tag{14}$$

If Case B is true, then $J_1(\gamma_{0M}(u_{1M}), u_{1M}) = M$ from the construction of γ_{0M} in (13). In this case, (u_0^*, u_1^*) is inducible only if [according to (5)]

$$J_0(u_0^*, u_1^*) \geq \sup J_0(\gamma_{0M}(u_{1M}), u_{1M}) \triangleq b, \tag{15}$$

where the supremum is taken over all solutions of (**a) which achieve its minimum. This implies that Lemma 2 is true.

Lemma 2: If Case B is true, then IR $= D_0 U D_2$, where

$$D_2 = \{(u_0, u_1) | J_1(u_0, u_1) = M \quad \text{and } J_0(u_0, u_1) \geq b\}. \tag{16}$$

If Case C is true, then for each $u_1 \in U_1$ there exists a $u_0 \in U_0$ such that $J_1(u_0, u_1) > M$. To see this, suppose it is not true, then there exists a u_1' such that $J_1(u_0, u_1') \leq M$ for all $u_0 \in U_0$. This implies that either 1) there exists a $u_0 \in U_0$ such that $J_1(u_0, u_1') = M$, then we have $u_1' \in U_{1a}$; or 2) $J_1(u_0, u_1') < M$ for all $u_0 \in U_0$, then we have $u_1' \in \hat{U}_1$. Both cases lead to contradiction. As a result, any point (u_0^*, u_1^*) on the $J_1 = M$ boundary is inducible. We thus obtain the result.

Lemma 3: If Case C is true, then IR $= D_1$, where

$$D_1 = \{(u_0, u_1) | J_1(u_0, u_1) \leq M\}. \tag{17}$$

The above results can be summarized by the following theorem.

Theorem 1: A Stackelberg strategy exists iff the problem

$$\min_{(u_0, u_1) \in \text{IR}} J_0(u_0, u_1), \tag{18}$$

has a solution, where IR is defined by (14)–(17) for various cases.

Theorem 2: An ϵ-Stackelberg strategy always exists, provided that J_0^s is bounded.

Proof: Under the boundedness condition, and for a given $\epsilon > 0$, from (7), there always exists a point $(u_0, u_1) \in$ IR such that

$$|J_0(u_0, u_1) - J_0^s| < \epsilon.$$

Since this $(u_0, u_1) \in$ IR is inducible, an ϵ-Stackelberg solution exists.

VI. Concluding Remarks

In this note, necessary and sufficient conditions for two-person, single-stage games are derived by using the concept of inducible region. Since the inducible region concept is derived directly from the definition, the inducible region in strategy space always exists regardless of whether the problem is deterministic or stochastic, single-stage or multistage, one-follower or multifollower as discussed in [12]. Thus, it is expected that the concept can be carried through for general problems. Preliminary results are encouraging (e.g., [10]) and will be reported in forthcoming papers.

Those who are familiar with Tolwinski's results in [7] might see the mathematical resemblance between his results and ours. According to Tolwinski [11], ideas relevant to the concept of inducible region have appeared in the Russian literature. However, they have not been introduced formally. He is the first researcher to obtain a region which is close to the inducible region for the problem considered. We define the inducible region formally and delineate it step by step from the definition. It turns out that his region in [7, eq. (11)] and the inducible region differ from each other only on the $J_1 = M$ boundary. However, for more general problems such as the three-level games considered in his paper, his region defined [7, eq. (40)] and the actual inducible region for that problem are very different. This point has been discussed in detail in [13].

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On the Efficiency of Output-Error Estimators

PAUL KABAILA

***Abstract*—In this correspondence we consider a system with rational transfer function from input to output where the transfer function has p parameters. It is proved in [1] that when the input is a sum of sinusoids and has a two-sided line spectrum consisting of p distinct frequencies, then the output-error estimator and the prediction-error estimator are equally efficient. This result, which is at first sight a surprising one, is given an intuitively pleasing proof in this correspondence. This proof is based on the algorithm of [2].**

I. INTRODUCTION

Let $\{u_t\}$ be the input, $\{y_t\}$ the output, and $\{\epsilon_t\}$ the driving (white) noise for the following single-input, single-output system:

$$y_t = z^s A(a,z) B(b,z)^{-1} u_t + H(\delta,z)\epsilon_t$$

where

$$A(a,z) := a_0 + a_1 z + \cdots + a_q z^q$$

$$a := [a_0, \cdots, a_q]'$$

$$B(b,z) := 1 + b_1 z + \cdots + b_{p-q-1} z^{p-q-1}$$

$$b := [b_1, \cdots, b_{p-q-1}]'$$

$$H(\delta,z) := \sum_{u=0}^{\infty} h_u(\delta) z^u$$

$$\delta := [\delta_1, \cdots, \delta_r]', \qquad \delta \in C.$$

a, b, δ are the true parameter values, s is assumed known. Here, $p := q$ is to be interpreted as meaning that p is defined by the expression q and x' denotes the transpose of x. We use a common form of notation in which z sometimes denotes the delay operator and sometimes a complex number. Suppose that a and b are unknown and that we wish to estimate these on the basis of $\cdots, u_0, \cdots, u_{n-1}$ and $y_0, \cdots, y_{n-1}$. In other words, we wish to estimate the transfer function from input to output.

Let us denote the output-error estimators of a and b by $\hat{a}_n$ and $\hat{b}_n$, respectively. Also let $\hat{a}_n = [\hat{a}_{n0}, \cdots, \hat{a}_{nq}]$, and $\hat{b}_n = [\hat{b}_{n1}, \cdots, \hat{b}_{n,p-q-1}]'$. As in [1] and [2] $\hat{a}_n$ and $\hat{b}_n$ are defined to be minimizing values of the following quantity subject to $a \in A$ and $b \in B$:

$$\frac{1}{n}\sum_{t=0}^{n-1}\left(y_t - z^s A(a,z) B(b,z)^{-1} u_t\right)^2.$$

Also, let us denote the prediction-error estimators of a, b, and δ by $\tilde{a}_n$, $\tilde{b}_n$, and $\tilde{\delta}_n$, respectively. Also, let

$$\tilde{a}_n = [\tilde{a}_{n0}, \cdots, \tilde{a}_{nq}]',$$

$$\tilde{b}_n = [\tilde{b}_{n1}, \cdots, \tilde{b}_{n,p-q-1}]',$$

and

$$\tilde{\delta}_n = [\tilde{\delta}_{n1}, \cdots, \tilde{\delta}_{nr}]'.$$

As in [1, sect. IV] $\tilde{a}_n$, $\tilde{b}_n$, and $\tilde{\delta}_n$ are defined to be minimizing values of the following quantity subject to $a \in A$, $b \in B$, and $\delta \in C$:

$$\frac{1}{n}\sum_{t=0}^{n-1}\left(\sum_{v=0}^{t-1} i_v(\delta) y_{t-v} - I(\delta,z) z^s A(a,z) B(b,z)^{-1} u_t\right)^2 \quad (1.1)$$

where

$$I(\delta,z) := \sum_{v=0}^{\infty} i_v(\delta) z^v := H(\delta,z)^{-1}.$$

In [1] are specified regularity conditions under which

$$\left.\begin{aligned} n^{1/2}\left([\tilde{a}_n', \tilde{b}_n']' - [a', b']'\right) &\xrightarrow{d} N(0,F) \\ n^{1/2}\left([\hat{a}_n', \hat{b}_n']' - [a', b']'\right) &\xrightarrow{d} N(0,G) \end{aligned}\right\} \quad (1.2)$$

when $\xrightarrow{d}$ denotes convergence in distribution. Furthermore, [1, Theorem 4.1] provides regularity conditions under which the following result holds.

Result $*$: If the input u_t is a sum of sinusoids and has a two-sided line spectrum consisting of p distinct frequencies, then $F = G$, i.e., the output-error and prediction-error estimators of $[a', b']'$ have the same efficiency.

At first sight this is a rather surprising result. That $G \geq F$ (i.e., $G - F$ is positive semidefinite) is well to be expected but a simple condition (as provided by Result $*$) for $F = G$ is somewhat unexpected. Unfortunately, the proof of [1, Theorem 4.1] does not help one's intuition much as regards Result $*$. Our aim is to prove Result $*$ (under the appropriate regularity conditions) in a manner which helps one's intuitive understanding of it. In the next section we will prove that (under the appropriate regularity conditions)

$$n^{1/2}(\tilde{a}_n - \hat{a}_n) \xrightarrow{p} 0, \qquad n^{1/2}(\tilde{b}_n - \hat{b}_n) \xrightarrow{p} 0 \quad (1.3)$$

when $\xrightarrow{p}$ denotes convergence in probability. Equation (1.3) implies that, asymptotically, $[\tilde{a}_n', \tilde{b}_n']'$ and $[\hat{a}_n', \hat{b}_n']'$ amount to the same estimator.

II. THE MAIN RESULT

For the statement of the main result we require the following assumptions. Note that $M < \infty$.

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