JOURNAL OF OPTIMIZATION THEORY AND APPLICATIONS: Vol. 55, No. 1, OCTOBER 1987

Pricing Problems with a Continuum of Customers as Stochastic Stackelberg Games[1]

P. B. LUH,[2] T. S. CHANG,[3] AND T. NING[4]

Communicated by J. B. Cruz, Jr.

Abstract. The pricing problem where a company sells a certain kind of product to a continuum of customers is considered. It is formulated as a stochastic Stackelberg game with nonnested information structure. The inducible region concept, recently developed for deterministic Stackelberg games, is extended to treat the stochastic pricing problem. Necessary and sufficient conditions for a pricing scheme to be optimal are derived, and the pricing problem is solved by first delineating its inducible region, and then solving a constrained optimal control problem.

Key Words. Stackelberg games, stochastic games, pricing problems, nonnested information, multiperson optimization.

1. Introduction

The problem of pricing concerns the design of pricing schemes so as to maximize certain criterion such as social welfare, profit, etc. The problem has been studied by economists (e.g., Refs. 1 and 2) and treated by using the calculus of variations with several problem transformations along the line.

[1] The research work reported here as supported in part by the National Science Foundation under Grant ECS-81-05984, Grant ECS-82-10673, and by the Air Force Office of Scientific Research under AFOSR Grant 80-0098.

[2] Associate Professor, Department of Electrical and Systems Engineering, University of Connecticut, Storrs, Connecticut.

[3] Assistant Professor, Department of Electrical and Computer Engineering, University of California, Davis, California.

[4] Assistant Professor, Department of Electrical and Computer Engineering, Trinity College, Hartford, Connecticut.

In this paper, we shall view the pricing problem from a different angle. We observe that a company has to design its pricing scheme and has to make it known to the customers. Knowing the pricing scheme, each customer then decides the amount of the goods to purchase, based on his valuation of the goods and the pricing scheme. Thus, in designing the pricing scheme, the company has to take the customers' reactions into account. This problem falls into a special class of multi-person decision problems, known as the *Stackelberg games* or *leader-follower problems* (Refs. 3-6), as the company (the leader) can design its pricing scheme and impose it on customers (followers).

The purpose of this paper is twofold. On the one hand, we interpret the pricing problem as a Stackelberg game. Therefore, we can use the inducible region concept, a notion recently developed for deterministic Stackelberg games (Refs. 6-8) and extended here to stochastic games, to provide a clear conceptual viewpoint of the problem. On the other hand, through the systematic procedure of the inducible region approach, we make clear what are the necessary and sufficient conditions for a pricing scheme to be optimal. A given problem can be solved by first delineating its inducible region, and then solving a constrained optimal control problem.

The problem formulation is given in Section 2. Since Spence's works are closely related to ours, his results are reviewed in Section 3. The inducible region concept for stochastic problems is introduced in Section 4. Our main results are then presented in Section 5.

2. Pricing Problems as Nonnested Stochastic Stackelberg Games

Consider a pricing problem where there is a company (the leader, DM0) that sells a product to a continuum of customers (followers). The customers are indexed by a parameter z, indicating individual's valuation of the goods. Assume that $z \in Z = [z_m, z_M]$ and is described by a probability density function $P_Z(z)$ known to the leader. A customer with parameter z is denoted as $DM1_z$. Let r_0 denote the company's pricing scheme with pricing differential being not allowed; i.e., the price a customer pays depends only on the amount of the goods he purchases, but not on the private parameter z. Therefore, r_0 can be a function of u_1 only. For example, if one selects a pure price system, $r_0(u_1) = pu_1$, for same price p. Consequently, if $DM1_z$ purchases $u_1 > 0$ units of the product, he pays $r_0(u_1)$ dollars. In this case, $DM1_z$'s net benefit can be represented as

$$J_1(u_1, z) = S(u_1, z) - r_0(u_1), \tag{1}$$

where $S(u_1, z)$ is $DM1_z$'s satisfaction in dollar value by consuming u_1 units

of the product. We assume that S is twice differentiable, with $S_{u_1 z} > 0$, so that a customer with higher valuation of the goods has higher marginal satisfaction at every level of u_1. The function $r_0(u_1)$ is assumed to be twice differentiable for $u_1 > 0$. We also assume that $r_0(0^-) = 0$; i.e., a customer pays nothing if he does not buy it. For a given r_0, $DM1_z$ decides an optimal u_1 to maximize J_1; i.e., he solves

$$\max_{u_1} \{S(u_1, z) - r_0(u_1)\}, \qquad \text{subject to } u_1 \geq 0. \tag{2}$$

The solution of (2) is denoted as $r_1(z)$. It represents $DM1_z$'s reaction to the given r_0. The price he has to pay for it is $u_0 = r_0(r_1(z))$. In this paper, we assume that $r_1(z)$ exists and is unique. We further assume that there exists a $z_d \in (z_m, z_M)$ such that $r_1(z) = 0$, for $z \leq z_d$, and $r_1(z)$ is differentiable for $z > z_d$. The parameter z_d can be interpreted as the entry point of the game, as will become clear later in Section 3.

The company's payoff function is described by

$$J_0 = \int_{z_m}^{z_M} \{L_0(r_0(u_1), r_1(z), z) P_Z(z)\}\, dz = E\{L_0\}. \tag{3}$$

J_0 can be the profit, a social welfare function, or any meaningful function from the company's viewpoint. For the moment, however, we shall not specify L_0 explicitly. Knowing the density function $P_Z(z)$ and the followers' rationale (2), DM0 wants to select a strategy r_0 from the admissible set Γ_0 to maximize J_0. That is, DM0 solves the following problem:

$$(P1) \quad \max_{r_0 \in \Gamma_0} E\{L_0(r_0(u_1), r_1(z), z)\},$$
$$\text{subject to the followers' problem (2).} \tag{4}$$

The sequence is shown in Fig. 1.

Note that the leader can be conceived of as having one aggregated follower with the private information z. Therefore, the problem can also be

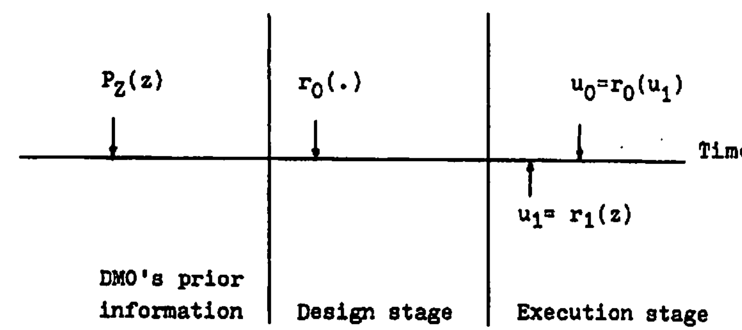

Fig. 1. Decision sequence of the leader and the follower.

regarded as a single-stage, one-leader-one-follower stochastic Stackelberg game with nonnested information structure (nonnested information structure is defined in Ref. 9).

3. Spence's Results

Since Spence's works in Ref. 1 are closely related to ours, we shall first paraphrase his relevant results. For a given r_0, a customer has to decide whether to buy the product or not. Since we assume that $r_0(0^-)=0$, the customer will not purchase any of the product (i.e., $u_1=0$) if

$$S(0,z)>\{S(u_1,z)-r_0(u_1)\}, \qquad \forall u_1>0. \tag{5}$$

Assume that z_d is the largest value of z such that $u_1=0$ for $z\le z_d$. That is, z_d is the entry point, and a customer will enter the market only if $z>z_d$. As assumed, there exists an entry point $z_d\in(z_m,z_M)$. The case where all customers participate can be treated in a similar way, and is thus omitted. Interested readers are referred to Ref. 10.

For a customer with $z>z_d$, the first-order necessary condition of his optimization problem is

$$dJ_1/du_1=0,$$

or

$$S_{u_1}=dr_0/du_1, \qquad \text{for } z>z_d. \tag{6}$$

Its solution leads to the follower's reaction $r_1(z)$. Substituting $r_1(z)$ into J_1, and differentiating $J_1(r_1(z),z)$ with respect to z, we obtain

$$\begin{aligned} dJ_1/dz &= S_z+(S_{u_1})\,dr_1/dz-(dr_0/du_1)\,dr_1/dz \\ &= S_z+(S_{u_1}-dr_0/du_1)\,dr_1/dz \\ &= S_z, \qquad \text{for } z>z_d. \end{aligned} \tag{7}$$

Integrating (7), we then have

$$\hat{J}_1(z)=J_1(r_1(z),z)=\int_{z_d}^{z} S_z\,dz'+C_1, \tag{8}$$

where

$$C_1=J_1(r_1(z_d^+),z_d^+)$$

is the integration constant. From (1), we obtain

$$\begin{aligned} r_0(r_1(z)) &= S(r_1(z),z)-J_1(r_1(z),z) \\ &= S(r_1(z),z)-\int_{z_d}^{z} S_z(r_1(z'),z')\,dz'-C_1, \qquad \text{for } z>z_d. \end{aligned} \tag{9}$$

Substituting (9) into (3), we have

$$J_0=\int_{z_m}^{z_d} L_0(0,0,z)P_Z(z)\,dz+\int_{z_d}^{z_M} L_0(r_0(r_1(z)),r_1(z),z)P_Z(z)\,dz. \tag{10}$$

Thus, J_0 can be represented as a function of $r_1(z)$. If we solve

$$(P2) \quad \max_{r_1(z)} J_0(r_1(z)),$$

then the optimal value of $(P2)$ is an upper bound to the leader's optimal payoff. Assume that the solution of $(P2)$ exists, as denoted by $r_1^0(z)$. Spence further assumes that the inverse function of $u_1=r_1^0(z)$ exists, as denoted by $z=r_1^{0-1}(u_1)$. From (6), we can then obtain

$$r_0(u_1)=\int_{u_{1d}}^{u_1} S_{u_1}(u_1', r_1^{0-1}(u_1'))\,du_1'+C_2, \tag{11}$$

where

$$u_{1d}=r_1^0(z_d^+).$$

If this $r_0(u_1)$ can be used to induce $r_1^0(z)$, then the leader's upper bound is achieved, and the problem is solved.

The difficulty of this approach is that some $r_1(z)$, although desirable from the leader's viewpoint, might not be realizable. That is, there might not exist an r_0 to induce it. The reason is that, in problem $(P2)$, J_0 is maximized over the set of all the r_1's. If the resulting solution happens to be inducible, it is alright. If it is not, then the solution cannot be used and the approach fails. By examining the follower's second-order condition, Spence suggested that the problem in that case must be redone with some additional constraint.

4. Inducible Region for Stochastic Games

In this section, we shall extend the inducible region concept of deterministic Stackelberg games (Refs. 6-8) to stochastic settings and shall delineate it for the pricing problem under consideration.

For a given r_0, the solution of (2) yields $r_1(z)$. We say that this r_1 is induced by r_0 and (r_0,r_1) is an *inducible pair of strategies*. For a different DM0's strategy, the followers' reactions will be changed correspondingly, and we have another inducible pair of strategies. The *inducible region in the* *strategy space* (IRSS) is defined as the collection of all the inducible pairs of strategies, i.e.,

$$\text{IRSS} = \{(r_0, r_1): (r_0, r_1) \text{ is an inducible pair of strategies}\}. \tag{12}$$

Since any realizable strategy pair must belong to IRSS, Problem $(P1)$ is equivalent to the following problem:

$$(P3) \quad \max_{(r_0, r_1) \in \text{IRSS}} E\{L_0(r_0, r_1, z)\}.$$

Let us now consider a pair $(r_0, r_1) \in$ IRSS. DM1_z's decision is given by $u_1 = r_1(z)$. The corresponding u_0 is determined by

$$r_0(z) = r_0(r_1(z)) = f_0(z),$$

where $f_0(z)$ is defined as the composite function of r_0 and r_1 that maps Z into U_0. The pair $(u_0 = f_0(z), u_1 = r_1(z))$ can be regarded as the outcome for DM1_z for this r_0, i.e., the price he paid and the amount of the goods he got. For a follower with parameter z', the corresponding outcome is given by $(f_0(z'), r_1(z'))$. Thus, across the population, the outcomes are described by the pair of functions (f_0, r_1). We then define the *inducible region in the decision space* (IRDS) as

$$\text{IRDS} = \{(f_0, r_1): \exists (r_0, r_1) \in \text{IRSS, s.t. } f_0(z) = r_0(r_1(z)), \forall z \in Z\}. \tag{13}$$

It is clear that we have the following lemma.

Lemma 4.1. Problem $(P1)$ has a solution if and only if Problem $(P4)$ defined below has a solution:

$$(P4) \quad \max_{(f_0, r_1) \in \text{IRDS}} E\{L_0(f_0, r_1, z)\},$$

where appropriate substitutions of r_0 by f_0 in L_0 are assumed.

5. Delineation of the Inducible Region

To delineate the inducible region, we examine an element $(f_0, r_1) \in$ IRDS. We first note that the relationship between f_0 and r_1 has been described by Spence's results as given in (9), i.e.,

$$f_0(z) = S(r_1(z), z) - \int_{z_d}^{z} S_z \, dz' - C_1. \tag{14}$$

To determine the integration constant C_1, we note that

$$\hat{J}_1(z_d^-) = S(0, z_d^-), \tag{15}$$

$$\hat{J}_1(z_d^+) = S(u_{1d}, z_d^+) - r_0(u_{1d}) = C_1. \tag{16}$$

Since $\hat{J}_1$ is continuous at z_d [as can be seen from (5)], and since S is continuous in its arguments, we have

$$C_1 = S(0, z_d) = S(u_{1d}, z_d) - r_0(u_{1d}). \tag{17}$$

To obtain further conditions on r_1, we consider the follower's second-order necessary condition

$$d^2J_1/du_1^2 = (S_{u_1u_1} - d^2r_0/du_1^2) \leq 0, \qquad \text{for } z > z_d. \tag{18}$$

By taking total derivative on both sides of (6) with respect to z, we have

$$(S_{u_1u_1})\, dr_1/dz + S_{u_1z} = (d^2r_0/du_1^2)\, dr_1/dz,$$

or

$$\{S_{u_1u_1} - (d^2r_0/du_1^2)\}\, dr_1/dz = -S_{u_1z}, \qquad \text{for } z > z_d. \tag{19}$$

Since $S_{u_1z} > 0$ by assumption, the right-hand side of (19) is not zero. Therefore, neither $S_{u_1u_1} - d^2r_0/du_1^2$ nor dr_1/dz is zero. One immediate observation following from (19) is that, assuming the differentiability of $r_1(z)$, the follower's second-order necessary condition actually implies

$$d^2J_1/du_1^2 = \{-S_{u_1z}/(dr_1/dz)\} < 0, \qquad \text{for } z > z_d, \tag{20}$$

and

$$dr_1/dz > 0, \qquad \text{for } z > z_d. \tag{21}$$

It is obvious that (6), (20), (21) are sufficient conditions for an r_1 to be optimal. Furthermore, based on the assumptions of $S_{u_1z} > 0$ and of the differentiability of $r_1(z)$ for $z > z_d$, Eqs. (6), (20), (21) turn out also to be necessary. Consequently, we have the following lemma.

Lemma 5.1. If $S_{l_1z} > 0$ and $r_1(z)$ is differentiable for $z > z_d$, then $r_1(z)$ is an optimal solution for DM1_z if and only if (6), (20), (21) are satisfied.

We then have the following theorem.

Theorem 5.1. The inducible region IRDS is delineated by

$$\text{IRDS} = \{(f_0, r_1): f_0 \text{ is given by (14); } dr_1/dz > 0, \text{ for } z > z_d;$$
$$r_1(z) = 0, \text{ for } z < z_d,\ z_d \in (z_m, z_M)\}. \tag{22}$$

Proof. We have to show that, for any (f_0, r_1) IRDS of (22), there exists an r_0 which induces r_1. Note first that the function $u_1 = r_1(z)$ is invertible for $z > z_d$ since $dr_1/dz > 0$, as derived previously in Lemma 5.1. Let r_1^{-1} denote the inverse function of r_1 in this region. Substituting z in the first-order condition (6) by $r_1^{-1}(u_1)$, and integrating with respect to u_1 on both sides, we have

$$r_0(u_1)=\int_{u_{1d}}^{u_1}(S_{u_1}(u_1', r_1^{-1}(u_1'))\,du_1'+C_2, \tag{23}$$

where

$$C_2=r_0(u_{1d})$$

is the integration constant and can be obtained from(17),

$$C_2=r_0(u_{1d})=S(u_{1d}, z_d)-S(0, z_d). \tag{24}$$

It is easy to see that r_0 of (23) induces r_1 by noting that the follower's necessary and sufficient conditions ((6), (20), (21)) are satisfied. Finally, to show that f_0 of (14) can also be obtained from $r_0(u_1)$ of (23), simply observe that the following equality holds:

$$dS=(S_z)\,dz+(S_{u_1})\,du_1. \tag{25}$$

After integration, we have

$$S-\int_{z_d}^{z}(S_z)\,dz'=\int_{u_{1d}}^{u_1}(S_{u_1})\,du_1'+C_3. \tag{26}$$

For the second part of the proof, it is easy to see that an arbitrary r_0 and its induced r_1 result in a pair $(f_0(z), r_1)$ belonging to IRDS. Consequently, the proof is completed. □

From Lemma 4.1 and Theorem 5.1, we then obtain the following results.

Corollary 5.1. Problem $(P1)$ has a solution if and only if Problem $(P5)$ defined below has a solution:

$(P5)$ $\max_{r_1} E\{L_0(r_1, z)\}$,

subject to $dr_1/dz>0$, for $z>z_d$,

and $r_1(z)=0$, for $z\leqslant z_d$, $z_d\in(z_m, z_M)$,

where appropriate substitutions of r_0 by f_0 of (14) are assumed.

Problem $(P1)$ is generally an intractable problem, due to the presence of the composite function $r_0(r_1(z))$ and the nontested information structure. Problem $(P2)$ is the maximization of J_0 with respect to a single function r_1. However, there is no guarantee that the resulting r_1 would be inducible. In $(P5)$, J_0 is optimized over the set of all inducible r_1's, and the conditions for the existence of an optimal r_1 (and therefore for the existence of an optimal r_0) are necessary and sufficient. Problem $(P5)$ is an optimal control-like problem with an inequality constraint. Although its solution is nontrivial and subject to further investigation, we have converted the original intractable problem $(P1)$ to the comparatively much simpler one $(P5)$. As suggested by one of the reviewers, the constraint $dr_1(z)/dz>0$ can be eliminated by introducing a control $v(z)$ and setting

$$r_1(z)=\int_{z_d}^{z}\exp(v(t))\,dt+C. \tag{27}$$

Once $(P5)$ is solved, its result r_1^s is then the optimal inducible r_1. The optimal pricing strategy can be constructed according to (23). A final note is that, in order to use this approach, it is not necessary to have the leader's cost function in the form of (3), as illustrated by the following example.

Example 5.1. *Welfare Maximizing Company.* Consider a specific pricing model where the customers' satisfaction function is given by

$$S(u_1, z)=(w_1/2)u_1^2+w_2u_1z+(w_3/2)z^2. \tag{28}$$

It is assumed that

$$W_1<0 \quad\text{and}\quad S_{u_1z}=w_2>0.$$

Let K denote the total number of customers in the population. The company has a quadratic production cost,

$$C=a_0+a_1u_1^T+a_2(u_1^T)^2, \tag{29}$$

where u_1^T denotes the total amount of the product sold by the company, i.e.,

$$u_1^T=K\int_{z_m}^{z_M}r_1(z)P_Z(z)\,dz. \tag{30}$$

The revenue collected by the company is denoted by R,

$$R=K\int_{z_m}^{z_M}r_0(r_1(z))P_Z(z)\,dz. \tag{31}$$

The profit is given by $R-C$. On the other hand, the aggregate welfare gain of the customers (consumer surplus, CS) is

$$\text{CS}=K\int_{z_m}^{z_M}(S(r_1(z), z)-r_0(r_1(z))P_Z(z)\,dz. \tag{32}$$

One measure of the social welfare gain is the sum of the company's profit and the aggregate welfare gain of the customers, i.e.,

$$\begin{aligned} J_0&=\text{CS}+(R-C)\\ &=K\int_{z_m}^{z_M}S(r_1(z), z)P_Z(z)\,dz-a_0-a_1u_1^T-a_2(u_1^T)^2. \end{aligned} \tag{33}$$

We shall assume that $z_d>z_m$.

Following the methods developed in the previous section, and using the calculus of variations in solving $(P5)$, we obtain (details are provided in the appendix).

$$r_1^s(z) = \begin{cases} -(w_2/w_1)z + g(z_d)/w_1, & \text{for } z > z_d, \\ 0, & \text{otherwise,} \end{cases} \tag{34}$$

where

$$g(z_d) = (a_1 w_1 - 2a_2 K w_2 \hat{z})/(w_1 - 2a_2 K h(z_d)), \tag{35}$$

$$\hat{z} = \int_{z_d}^{z_M} z P_Z(z)\, dz, \qquad h(z_d) = \int_{z_d}^{z_M} P_Z(z)\, dz. \tag{36}$$

We observe that

$$dr_1^s/dz > 0, \qquad \text{for } z > z_d.$$

The first-order necessary condition for z_d yields the following implicit equation

$$z_d = (a_1 w_1 - 2a_2 K w_2 \hat{z})/w_2(w_1 - 2a_2 K h(z_d)), \tag{37}$$

and z_d and C_2 are related by

$$C_2 = -\{g^2(z_d) - w_2^2 z_d^2\}/2w_1. \tag{38}$$

For z uniformly distributed between 0 and 1, z_d can be solved explicitly as

$$z_d = \{a_1 w_1 - 2a_2 K w_2\}/\{w_2(w_1 - 2a_2 K)\}.$$

Finally, from (24), we have

$$r_0^s(u_1) = \begin{cases} g(z_d)u_1 + (g^2(z_d) - w_2^2 z_d^2)/2w_1, & \text{for } u_1 > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{39}$$

The details are given in the appendix.

6. Concluding Remarks

Pricing problems are formulated as stochastic Stackelberg games with nonnested information structures and are studied by using the inducible region concept. It was thought that such problems were extremely difficult to solve, and their inducible regions were not clearly defined. However, by exploiting the special structure of the model, it turns out that the leader is capable of recovering the follower's private information through their decisions; therefore, we are able to delineate the complete inducible region for the pricing problems under consideration. The optimal $r_1(z)$ can then be obtained by solving an optimal control-like problem. The corresponding optimal pricing scheme can also be constructed. The methodology can be extended to multistage pricing problems of the like, and explicit results are obtained for a two-stage pricing problem in Ref. 10.

7. Appendix: Details of the Welfare Maximizing Example

A customer with $z > z_d$ has a net gain

$$J_1(u_1, z) = S(u_1, z) - r_0(u_1). \tag{40}$$

On the other hand, a customer with $z < z_d$ has a net gain

$$J_1^0 = (w_3/2)z^2. \tag{41}$$

Since

$$J_1^0 = J_1(u_1, z), \qquad \text{at } z = z_d,$$

we have

$$r_0(u_{1d}) = S(u_{1d}, z_d) - w_3 z_d^2/2 = C_2. \tag{42}$$

We shall then treat z_d (instead of C_2) as an independent variable to be optimized. Once z_d is determined, C_2 can be calculated according to (42).

The company's payoff can be rewritten as

$$J_0 = K \int_{z_m}^{z_d} w_3 z^2/2 P_Z(z)\, dz + K \int_{z_d}^{z_M} S(u_1(z), z) P_z(z)\, dz - a_0 - a_1 u_1^T - a_2 (u_1^T)^2. \tag{43}$$

To maximize J_0, the company solves

$$\max_{z_d, u_1} \{J_0\}, \quad \text{subject to } du_1/dz > 0 \text{ and } z_d \in [z_m, z_M]. \tag{44}$$

Using the calculus of variations, we shall find the optimal u_1. The first-order necessary condition is

$$d(J_0(u_1 + \epsilon h))/d\epsilon = 0, \qquad \text{at } \epsilon = 0, \tag{45}$$

where h is a variation in u_1 and ϵ is a small number. We have

$$K \int_{z_d}^{z_M} \{S_{u_1} - a_1 - 2a_2 u_1^T\} h P_Z(z)\, dz = 0, \tag{46}$$

or

$$w_1 u_1 + w_2 z - a_1 - 2a_2 u_1^T = 0. \tag{47}$$