Load Adaptive Pricing: An Emerging Tool for Electric Utilities

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Abstract—Motivated by the importance of the peak load problem faced by the electric utility industries; in this paper we analyze several different electricity pricing schemes from a game theoretic point of view. Recognizing the limit of the traditional peak load pricing formulation and the persuasive breakthroughs in microelectronic technology, we introduce a philosophy in which supply and demand respond to each other through prices and consumptions, and the utility company sells power at "real-time" rates. We call it *load adaptive pricing*. By itself, the concept of load adaptive pricing is not new. The contribution of this paper is the formulation and resolution of this idea as a closed-loop dynamic Stackelberg game problem. The central part of this problem is how to choose appropriate incentives (i.e., pricing strategies) so that customers can be induced to behave cooperatively and thus achieve the team optimum. In this paper, the load adaptive pricing problem is solved for a particular producer/consumer model. We demonstrate that it is possible for the utility company to induce the customer to behave cooperatively to achieve the team optimum. We also show that in steady state, our solution converges and the system is stable.

I. Introduction

A. The Peak Load Problem and Current Practice

THE electric utilities[1] in the United States face the peak load problem. This problem arises because traditionally, the utilities are required to supply their customers instantaneously with whatever amount of power they wish to purchase at whatever time they desire, with prespecified prices which are independent of the time of consumption. In exercising their preferences, customers have evolved electric power use patterns which vary not only periodically over the day, week, and year, but also depend significantly on some random elements such as weather, for example. In fact, with few exceptions, all system peaks in the U.S. are caused principally by air conditioning (summer peak) and space heating (winter peak). Thus, the utility must meet the demand due to a fluctuating load instantaneously, incurring idle capacity cost, while capacity expansions require a fairly long gestation period. Most utilities use additional peaking generators (which have low capital cost to offset their idle time, but high running cost) to satisfy the excess loads during the short duration peak periods. The unit cost of electricity production during the peak period is thus higher than that in the off-peak periods.

In recent years the high cost of electricity production, especially during peaks, and the concern for future energy resources have led utilities and regulatory agencies to move to policies that encourage conservation and more efficient use of production capacity. Toward these ends, the utility may either negotiate for direct control of some of the customer loads as with industrial users and/or design some forms of incentive rate structures, which may be time-dependent, as an indirect way to influence some of the customer loads. The latter is a variable pricing scheme for electricity consumption [19]. The *peak load pricing* scheme is a familiar example. In peak load pricing, a *cycle* (usually a day, a week, or a year) is divided into several *periods* (a period may range from several hours to seasons). The price of electricity in each period reflects the estimated production costs for that period, and is required to be announced prior to the beginning of operation. The peak load pricing problem has been studied extensively by economists (for example: [6], [7], [10], [15], [17], [18], [22], [25], and [33]; other references can be found in [30] and [21]).

B. The Concept of Load Adaptive Pricing

From a game theoretic point of view, the utility company plays the role of a *leader*, and customers play roles of *followers* in the variable pricing framework. For a given pricing strategy, each customer determines his optimal consumption strategy which is reflected in the demand curve. The utility company foresees these reactions and decides the optimal pricing strategy. Thus, pricing problems are actually Stackelberg games.

However, in peak load pricing, the utility company is not a "powerful" leader. On the one hand, it needs to announce all the time-dependent prices prior to the beginning of operation, after which it is expected to persist without change for many cycles. (In the parlance of control theory, this is called open-loop control.) Price cannot respond to "real-time" loads caused by random events such as weather and outage under this setup. On the other hand, the utility

Manuscript received October 27, 1980; revised March 27, 1981 and September 17, 1981. Paper recommended by D. A. Hanson, Chairman of the Economic and Social Systems Committee. This work was supported by the Department of Energy under Contract ET-78-C-01-3252 and the National Science Foundation under Grants ENG 78-15231 and ECS 8105984.

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[1]The problem addressed in this note is common to many other public enterprises as well (telecommunications, for example), but we shall focus the discussion on electric utilities to keep the terminology simple and unambiguous.

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company bears almost all the burden concerning system operation, planning, integrity, and environmental impact. Customers are, in general, not supplied with any system demand/supply information. It will not be very effective for the utility company to induce a particular behavior on the part of customers without providing them with some kind of demand/supply information.

In recognizing the limit of the peak load pricing formulation and the persuasive breakthrough of microelectronic technology, we foresee a new pricing philosophy emerging in which supply and demand respond to each other through prices and consumptions. We call it *load adaptive pricing*. In load adaptive pricing, the utility company does not announce prices for all cycles prior to the beginning of operation. On the contrary, it announces the *strategy* of pricing, i.e., the rules for how the prices will be determined. The exact price for each period is calculated at the beginning of that period according to the announced strategy and based on previous consumptions and realizations of random events. Supply and demand thus respond to each other through prices and consumption, and the prices are made to adapt to the load.

Throughout the discussion we shall use a simplified single-producer single-consumer model to bring out the main aspects of pricing schemes. It represents the situation where a utility company deals with a single large industrial customer, or an organized group of customers which has the intention of improving social welfare.

By itself the concept of load adaptive pricing is not new [34], [26]. *The contribution of this paper is to give for the first time a mathematical formulation and resolution of this idea as a closed-loop dynamic Stackelberg game*. (For recent developments in Stackelberg games, see [28], [29], [3], [4], [23], [24], [31], [13], [14], and [32].) It should also be emphasized at the outset that the purpose of this paper is not to formulate a comprehensive model of load management taking into account all the socioeconomic and technological constraints and dynamics of a "real world" power generation/distribution network.[2] What we have done is to choose a more narrowly specified set of issues, mainly economic (corresponding to "spot pricing" in [27]), to demonstrate how mathematical tools of optimization and game theory can be used to address and solve these issues; and to interpret the results. In terms of the larger comprehensive model, our results illustrate the general method of attack and the conceptual approach. For the narrowly defined problem, our results identify the relevant parameters and ideas involved. We are not suggesting that our solution is ready and can be applied "as is" to a real world utility pricing problem.

C. Outline of the Paper

In Section II, the distinction between peak load pricing and load adaptive pricing is made clear through the notion of information structure. The deterministic producer/consumer model, [12] and [20], is extended in Section III for the mathematical formulation of the load adaptive pricing problem. The optimal load adaptive pricing problem is then solved in Section IV. The asymptotic behavior of the system is examined in Section V.

II. Pricing Schemes and Information Structures

We shall model various pricing schemes using the notion of information structure. The *information structure* for a game characterizes the precise information each player has at every stage of the game. Different information structures permit different decision rules or strategies, lead to different interpretations, and yield different results.

Let x_0 be the initial information available to both the producer and the consumer; and let p_i, q_i, and ξ_i be, respectively, the price, demand, and the state of nature at period i. Let η_{pi} be the information available to the producer at the instant when he needs to decide p_i, and let η_{ci} be the information available to the consumer at the instant when he needs to decide q_i. In this section, for the sake of clarity and ease of exposition, we shall consider a single cycle which consists of two periods.

In peak load pricing, the producer is required to declare the prices prior to the beginning of the cycle. For the producer we have the information structure

$$\eta_{p1}=\eta_{p2}=(x_0).$$

In most of the peak load pricing literature, the electricity consumed in n different periods was treated as if it had been n different commodities consumed in a single period. Consequently, consumption of the second period is not directly affected by what happened in the first period. The situation is modeled by the following information structure:

$$\eta_{c1}=(x_0,p_1,p_2,\xi_1)$$
$$\eta_{c2}=(x_0,p_1,p_2,\xi_2).$$

To be more realistic, the consumer should be endowed with the following information structure:

$$\eta_{c1}=(x_0,p_1,p_2,\xi_1)$$
$$\eta_{c2}=(x_0,p_1,p_2,\xi_1,q_1,\xi_2).$$

That is, the consumer acts according to prices, previous consumption level, and current and past realizations of ξ.

[2] Recently, a good conceptualization of a comprehensive model of load management has been proposed in [27] and refined in [16] under the name "homeostatic utility control," as a novel approach to the control and economic operation of electric power systems. It includes three major concepts: spot pricing, microshedding, and decentralized dynamic control. Spot pricing depends on system supply-demand conditions and is set every 5 min. With microshedding, the utility commands a customer's computer to shed a certain percentage or amount of the customer's load; the customer's computer then decides which part of the load to shed. Decentralized dynamic controllers are activated by changes in the frequency (or voltage) of the electric power system above and below the standard 60 Hz (or 120 V) and provide short-term storage adaptable to the power system. Load adaptive pricing corresponds to "spot pricing" of the homeostatic control. The theoretical analysis of customer response to spot pricing was given in [5].

The information structure for flat rate pricing is the same as that for peak load pricing with an additional constraint: $p_1 = p_2$.

Load adaptive pricing has the following information structure:

$$\eta_{p1} = (x_0)$$
$$\eta_{c1} = (x_0, p_1, \xi_1)$$
$$\eta_{p2} = (x_0, p_1, \xi_1, q_1)$$
$$\eta_{c2} = (x_0, p_1, \xi_1, q_1, p_2, \xi_2).$$

The producer is required to set up p_1 prior to the beginning of the cycle. The consumer then decides q_1 according to x_0, p_1, and the realization of ξ_1. Based on the realization of ξ_1 and the consumption q_1, the producer decides p_2. Finally, q_2 is determined according to the current price p_2, current realization of ξ_2, and past history (x_0, p_1, ξ_1, and q_1). Fig. 1 summarizes the relationship among information structures and pricing schemes.[3]

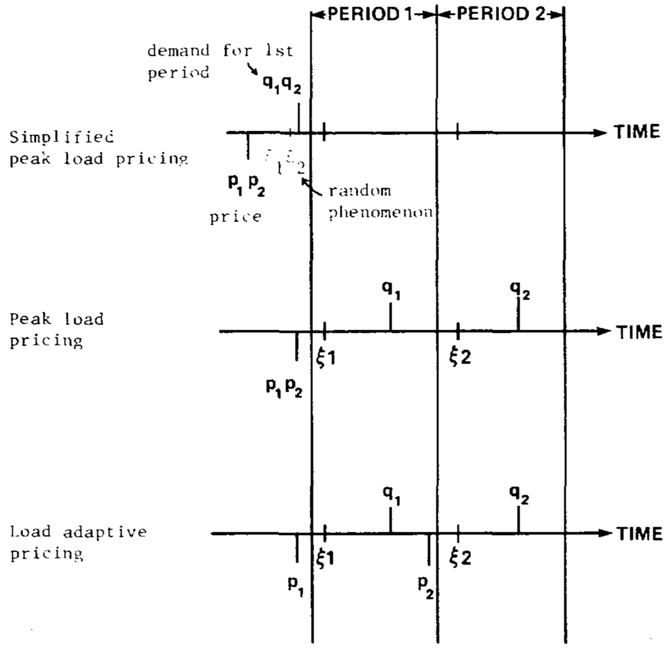

Fig. 1. Information structures and pricing schemes.

III. An Extended Producer/Consumer Model

We shall now extend the peak load pricing model of [12] and [20]. Consider a model of N cycles; each cycle in turn consists of M periods of equal duration. Each period is sufficiently short so that the demand for electricity within it can be assumed to be flat. For clarity of this discussion, however, M is restricted to be 2. The mth period of the nth cycle is written as period nm.

A. The Producer

As discussed earlier, the electric utilities in general have a variety of plant types to choose from. Base-load plants, such as nuclear and coal-steam, with high marginal capacity costs and low marginal operating costs, are used to furnish the base load of the demand. When the capacity of these plants is exceeded, supplementary plants are brought on line, such as oil-steam (for intermediate load) and internal combustion (for peak load). In [35] and [8] this multiplant situation was modeled as having a nondecreasing piecewise linear cost function (with each individual plant having a linear cost function). In [12] this piecewise linear cost function was further approximated by a quadratic cost function $C(q) = c_2 q^2 + c_1 q^1 + c_0$, where the coefficients c_i were derived from a minimum mean-square error fit to the piecewise linear curve. This quadratic cost function is adopted here with

$$C(q) = \frac{c}{2} \Sigma n, m\left(q_{nm}^2\right)$$

where c is a positive constant and q_{nm} is the electricity consumed in period nm.[4] The linear and constant terms have been dropped for simplicity. Note that the quadratic cost function captures the essence that the marginal cost of generation is an increasing function of output level.

B. The Consumer

The consumer is characterized by a satisfaction function $S(q)$, which is the preference function that gives in monetary units the level of satisfaction at the consumption level q. The basic form of the satisfaction function was derived from some assumptions about consumer behavior [12]; here, it is extended to the multicycle case, with the existence of both 1) uncertainties and 2) intercycle substitutions:

$$\begin{aligned} S(q) = \alpha\Big\{ & -\tfrac{1}{2}\Sigma n\Big[w_1(q_{n1} - \xi_{n1}\mathbf{q}_1)^2 + w_2(q_{n2} - \xi_{n2}\mathbf{q}_2)^2 \\ & + w_3(q_{n1} + q_{n2} - \xi_{n1}\mathbf{q}_1 - \xi_{n2}\mathbf{q}_2)^2\Big] \\ & - \tfrac{1}{2}\Sigma_{n=2}^{N} w_4\big(q_{n1} + q_{n2} + q_{n-1,1} \\ & + q_{n-1,2} - \xi_{n1}\mathbf{q}_1 - \xi_{n2}\mathbf{q}_2 \\ & - \xi_{n-1,1}\mathbf{q}_1 - \xi_{n-1,2}\mathbf{q}_2\big)^2\Big\} \end{aligned} \tag{1}$$

[3] Note that in this formulation we have a nested information structure in the sense that all succeeding decision makers know what the previous decision makers knew. This is not too far from real since most of the relevant uncertainties such as weather are observable by everyone. Advancements in communication and microelectronic technology will also make the utility company and customers better informed about relevant happenings. However, more research is needed for cases with nonnested information structure.

[4] Since we assumed all periods are of equal duration and the demand within each period is flat, the energy (kWh) consumed in a period is the power (kW) demanded in that period times the length of the period (h). Thus, in our mathematical formulation we shall not distinguish between power and energy. Note that the formulation can be extended to the case with periods of unequal duration without much difficulty [20]. In this case, the generating cost for a period is the length of the period times a quadratic function of demand in that period.

where w_i, $i=1,2,3,4$ are nonnegative constants and at least one of them is positive; $\boldsymbol{q}_j$ and α are positive constants; and ξ_{nm} are unit mean independent Gaussian random variables.[5]

In $S(q)$, $\xi_{n1}\boldsymbol{q}_1$ is the ideal consumption level for period $n1$ when all the prices are zero. It is a random variable and captures the random nature of the demand. The first term in $S(q)$, $-\frac{1}{2}w_1(q_{n1}-\xi_{n1}\boldsymbol{q}_1)^2$, reflects the consumer's desire to maintain the consumption schedule q_{n1} close to $\xi_{n1}\boldsymbol{q}_1$ with the priority w_1 in mind. The second term in $S(q)$ reflects the similar desire for period 2. The third term reflects the desire to keep the total consumption of the nth cycle $(q_{n1}+q_{n2})$ close to the ideal total $(\xi_{n1}\boldsymbol{q}_1+\xi_{n2}\boldsymbol{q}_2)$. Thus, if the price in the first period is increased drastically, the consumer may reduce the consumption and yet try to make up by consuming more in the second period (within the same cycle)—*intracycle substitution*. In the same spirit, the last term represents *intercycle substitution*. Note that intercycle substitutions are assumed to exist only between adjacent cycles. The third term tries to capture the essence of real-life consumption habits. If necessary, the model can be extended to incorporate intercycle substitutions across several cycles. Finally, the positive constant α translates the satisfaction of electricity consumption into monetary units. For simplicity we assume that α equals one.

All the observations are assumed to be noise-free, and both players have perfect memory.[6] The information structure is assumed to be nested [11]:

$$\begin{aligned}
\eta_{p,11}&: \quad \phi\\
\eta_{c,11}&: \quad p_{11}, k_{11}, \xi_{11}\\
\eta_{p,12}&: \quad p_{11}, k_{11}, \xi_{11}, q_{11}\\
\eta_{c,12}&: \quad p_{11}, k_{11}, \xi_{11}, q_{11}, p_{12}, k_{12}, \xi_{12}\\
\eta_{p,n1}&: \quad \eta_{c,n-1,2}\\
\eta_{c,n1}&: \quad \eta_{c,n-1,2}, p_{n1}, k_{n1}, \xi_{n1}\\
\eta_{p,n2}&: \quad \eta_{c,n-1,2}, p_{n1}, k_{n1}, \xi_{n1}, q_{n1}\\
\eta_{c,n2}&: \quad \eta_{c,n-1,2}, p_{n1}, k_{n1}, \xi_{n1}, q_{n1}, p_{n2}, k_{n2}, \xi_{n2}.
\end{aligned}$$

C. The Pricing Rules

The producer levies a two-part tariff on the consumer, with p_{nm} and k_{nm} being, respectively, the unit price and the fixed charge for period nm. Actions taken by the producer (p,k), the consumer (q), and the state of nature (ξ) are shown in Fig. 2. Note that in this case the revenue equals

$$R(q)=\Sigma_{n,m}(p_{nm}q_{nm}+k_{nm})+k_{N+1,1}.$$

The producer needs the additional (the final) control $K_{N+1,1}$ to induce the consumer's last action (q_{N2}). This becomes irrelevant as N approaches infinity, as we shall see in Section V.

[5]The assumptions of unit mean and independence among random variables are made to keep the discussion simple. They can be relaxed without much difficulty.

[6]It turns out that both the producer and the consumer do not need any information that occurred two or more cycles ago. See Section IV.

D. Payoff Functions and the Stackelberg Game

An important concept in our formulation is the notion of economic surplus. As described in the survey paper by Currie *et al.* [9], economic surplus is the benefit derived in monetary terms when a consumer purchases goods from a producer. The benefit to the consumer and the producer are called the "consumer's surplus" (CS) and the "producer's surplus" (PS), respectively. In our case, the consumer's surplus is the difference between his level of satisfaction and what he pays for it, i.e., $CS(q)=S(q)-R(q)$. Since CS is a measure of the net benefit the consumer derives from consuming q, we define the consumer's payoff function as the expected value of CS, i.e.,

$$J_c=E[CS(q)]=E[S(q)-R(q)]=E[L_c].$$

The producer's surplus is defined here as the profit, i.e., $PS(q)=R(q)-C(q)$. Many electric utilities are profit maximizing companies subject to the regulation of having a "fair" return on the total capital investment (see [18]). On the other hand, publicly owned and operated utilities may consider the benefit to both the consumer and the producer. One objective function that is both meaningful and quantifiable is the sum of the consumer's surplus and the producer's surplus ([25], [8], and [21]). We thus define the producer's payoff function as the expected value of the sum of producer's surplus and consumer's surplus,[7] i.e.,

$$\begin{aligned}
J_p&=E[CS(q)+PS(q)]\\
&=E[(S(q)-R(q))+(R(q)-C(q))]\\
&=E[S(q)-C(q)]=E[L_p].
\end{aligned}$$

For any given pricing strategy, the consumer chooses a reaction strategy which maximizes J_c. Knowing the consumer's rationale, the producer wishes to announce a strategy such that with this strategy and the consumer's reaction to it, the producer's maximum payoff is achieved. Thus, the problem formulated is a stochastic closed-loop multistage Stackelberg problem. *The central part of the problem is how to choose (closed-loop) pricing strategies so that the consumer can be induced to behave cooperatively and thus achieve the social optimum.* In the next section, we shall first derive the team solution, and then find a pricing scheme such that the consumer can be induced to behave cooperatively as a team member.

IV. Optimal Strategies

The best solution the leader can possibly achieve in a Stackelberg game is the team optimum, which is defined as the optimal payoff when all the players work cooperatively

[7]The nature of the regulatory environment is crucial to the pricing strategies. The well-known Averch-Johnson effect says that if a company's allowable profit is based on the amount of investment, then there is an incentive for the company to overcapitalize [1]. This in turn, has various implications on pricing policies [2]. We recognize that the concept of load adaptive pricing could have a major impact on the basic regulatory attitudes and approaches. However, we shall not address this issue here.

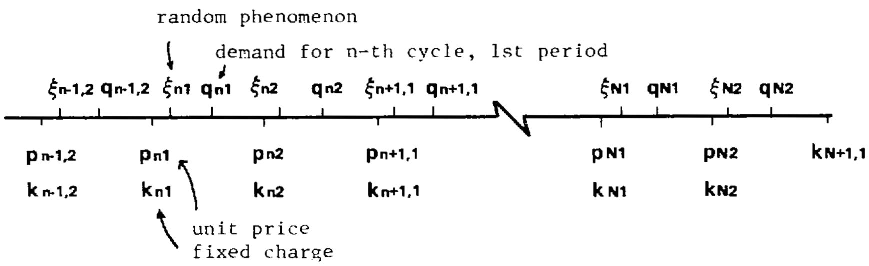

Fig. 2. Sequences of actions taken by decision makers in the load adaptive pricing formulation.

to optimize the leader's payoff under the same information structure. We shall investigate the team problem first.

A. Optimal Team Solution

Since p and k do not appear in J_p explicitly, the corresponding team problem here is to find q such that J_p is maximized. Also since the information structure is nested, the problem can be solved by using the usual dynamic programming method working backwards in time. It is easy to see that the necessary condition for q_{nm} to be optimal is

$$\frac{dE_{/cnm}[L_p]}{dq_{nm}} = E_{/cnm}\left[\frac{\partial S}{\partial q_{nm}} - cq_{nm}\right] = 0 \tag{2}$$

where $E_{/cnm}[\cdot]$ denotes the conditional expectation given the information of the consumer at period nm. Under the linear-quadratic-Gaussian assumption, we expect to find an affine solution [11]. Furthermore, since intercycle substitutions occur only between adjacent cycles, it is not difficult to see that the optimal team solution q^t_{nm} obtained from (2) will not be an explicit function of either $q_{n'm}$ or $\xi_{n'm}$ for $n' < n-1$. That is to say, in the team problem the consumer's action will be affected directly only by what happened in the current cycle and the one before that. Based on these observations and the particular format of L_p, the optimal team solution will be of the following form:

$$\begin{aligned} q^t_{n2} &= a_{n1}\xi_{n2}\mathbf{q}_2 + a_{n2}(q_{n1} - \xi_{n1}\mathbf{q}_1) \\ &\quad + a_{n3}(q_{n-1,1} + q_{n-1,2} \\ &\quad - \xi_{n-1,1}\mathbf{q}_1 - \xi_{n-1,2}\mathbf{q}_2) + a_{n4} \\ q^t_{n1} &= b_{n1}\xi_{n1}\mathbf{q}_1 + b_{n3}(q_{n-1,1} + q_{n-1,2} \\ &\quad - \xi_{n-1,1}\mathbf{q}_1 - \xi_{n-1,2}\mathbf{q}_2) + b_{n4} \end{aligned} \tag{3}$$

where a_{ni} and b_{nj} are constants determined by the following proposition. First some definitions:

$$\begin{aligned} x_{n1} &= a_{n3} + (1 + a_{n2})b_{n3} \\ x_{n2} &= w_2 + w_3 + 2w_4 + c + w_4 x_{n1} && \text{for } 2 < n < N \\ &= w_2 + w_3 + w_4 + c && \text{for } n = N+1 \\ &= w_2 + w_3 + w_4 + c + w_4 x_{n1} && \text{for } n = 2 \\ x_{n3} &= (w_1 + w_2 + 2c)x_{n2} - (w_2 + c)^2. \end{aligned}$$

Proposition 1: The team problem has a solution of the form (3), where the coefficients are determined by the following equations:

$$\begin{aligned} a_{n1} &= 1 - c/x_{n+1,2} && \forall n \\ a_{n2} &= -1 + (w_2 + c)/x_{n+1,2} && \forall n \\ a_{n3} &= -w_4/x_{n+1,2} && \text{for } 1 < n \leq N \\ &= 0 && \text{for } n = 1 \\ a_{n4} &= w_4[-a_{n+1,4} - (1 + a_{n+1,2})b_{n+1,4} \\ &\quad + (1 - a_{n+1,1})\mathbf{q}_2 + (1 + a_{n+1,2}) \\ &\quad \cdot (1 - b_{n+1,1})\mathbf{q}_1]/x_{n-1,2} && \text{for } 1 \leq n < N \\ &= 0 && \text{for } n = N \\ b_{n1} &= 1 - cx_{n+1,2}/x_{n+1,3} && \forall n \\ b_{n3} &= -w_4(w_2 + c)/x_{n+1,3} && \text{for } 1 < n \leq N \\ &= 0 && \text{for } n = 1 \\ b_{n4} &= \{-w_4(w_2 + c)[a_{n+1,4} + (1 + a_{n+1,2})b_{n-1,4}] \\ &\quad + w_4(w_2 + c)(1 + a_{n+1,2})(1 - b_{n+1,2})\mathbf{q}_1 \\ &\quad + [w_4(w_2 + c)(1 - a_{n+1,1}) \\ &\quad + c(x_{n+1,2} - w_2 - c)]\mathbf{q}_2\}/x_{n+1,3} \\ &&& \text{for } 1 \leq n < N \\ &= c(w_3 + w_4)\mathbf{q}_2/x_{n-1,3} && \text{for } n = N. \end{aligned}$$

Furthermore,

$$\begin{aligned} &-1 < x_{n1} \leq 0 \\ &x_{n2} > (w_2 + w_3 + c) > 0 \\ &x_{n3} > [(w_1 + c)(w_2 + c) + w_3(w_1 + w_2 + 2c)] > 0 \\ &\qquad \text{for } 2 \leq n \leq N+1. \end{aligned}$$

The lengthy, but straightforward proofs of this and all the following propositions will not be included here. They will be made available separately, if requested by the reader.

Although the above formulas for x_{ni}, a_{nj}, and b_{nk} seem quite complicated, it is only important at this point to recognize that the team problem has an affine solution, and the coefficients can be precalculated. The signs of x_{ni} indicate that

$$\begin{aligned} &1 > a_{n1} > 0, \quad a_{n2} < 0, \quad a_{n3} \leq 0 \\ &1 > b_{n1} > 0, \quad b_{n3} \leq 0. \end{aligned}$$

$a_{n1} > 0$ says that if the ideal consumption $\xi_{n2}\mathbf{q}_2$ is high for period $n2$, so is q^t_{n2}. The negativity of a_{n2} makes it clear that if the deviation from the ideal consumption in period $n1$, $(\xi_{n1}\mathbf{q}_1 - q_{n1})$ is large, then q^t_{n2} will be high in order to make up for the difference. $a_{n3} \leq 0$, $b_{n1} > 0$, and $b_{n3} \leq 0$ can be interpreted in the same fashion. Thus, the results we have are intuitively appealing.

B. Incentive Pricing Scheme

For any given pricing scheme, the consumer finds optimal consumption strategies q^*_{nm} such that J_c is maximized. The goal of the producer is to find a particular pricing scheme such that the consumer can be induced to act cooperatively, i.e., $q^*_{nm} = q^t_{nm}$, to achieve the team optimum. Due to the fact that q^t_{nm} is an explicit function of information in the nth and $n-1$th cycles only, we shall consider functions of the following form:

$$\begin{aligned}
&p_{n1}(\xi_{n-1,1}, q_{n-1,1}, \xi_{n-1,2}, q_{n-1,2})\\
&k_{n1}(\xi_{n-1,2}, q_{n-1,2})\\
&p_{n2}(\xi_{n-1,1}, q_{n-1,1}, \xi_{n-1,2}, q_{n-1,2}, \xi_{n1}, q_{n1})\\
&k_{n2}(\xi_{n1}, q_{n1}).
\end{aligned}$$

From the consumer's viewpoint, for any given pricing scheme of the above form he faces a one-person optimization problem. The necessary conditions for this problem can be obtained by using the dynamic programming method and are given by the following proposition.

Proposition 2: The necessary conditions for the consumer are

Period n2:

$$\begin{aligned}
\frac{dE_{/cn2}[L_c]}{dq_{n2}} &= E_{/cn2}\left[\frac{\partial L_c}{\partial q_{n2}} + \frac{\partial L_c}{\partial k_{n+1,1}}\frac{\partial k_{n+1,1}}{\partial q_{n2}}\right.\\
&\qquad \left. + \frac{\partial L_c}{\partial p_{n+1,1}}\frac{\partial P_{n+1,1}}{\partial q_{n2}} + \frac{\partial L_c}{\partial P_{n+1,2}}\frac{\partial P_{n+1,2}}{\partial q_{n2}}\right]\\
&= 0.
\end{aligned}$$

Period n1:

$$\begin{aligned}
\frac{dE_{/cn1}[L_c]}{dq_{n1}} &= E_{/cn1}\left[\frac{\partial L_c}{\partial q_{n1}} + \frac{\partial L_c}{\partial k_{n2}}\frac{\partial k_{n2}}{\partial q_{n1}} + \frac{\partial L_c}{\partial P_{n2}}\frac{\partial P_{n2}}{\partial q_{n1}}\right.\\
&\qquad \left. + \frac{\partial L_c}{\partial P_{n+1,1}}\frac{\partial P_{n-1,1}}{\partial q_{n1}} + \frac{\partial L_c}{\partial P_{n-1,2}}\frac{\partial P_{n+1,2}}{\partial q_{n1}}\right]\\
&= 0. \qquad (4)
\end{aligned}$$

Now we want to find some p and k such that $q^*_{nm} = q^t_{nm}$. We shall start with simple functional forms. Let

$$\begin{aligned}
p_{n1} &= e_{n3}(q_{n-1,1} + q_{n-1,2}\\
&\qquad - \xi_{n-1,1}\boldsymbol{q}_1 - \xi_{n-1,2}\boldsymbol{q}_2) + e_{n4}\\
k_{n1} &= e_{n1}(\xi_{n-1,2} - 1)q_{n-1,2}\\
p_{n2} &= d_{n2}(q_{n1} - \xi_{n1}\boldsymbol{q}_1) + d_{n3}(q_{n-1,1} + q_{n-1,2}\\
&\qquad - \xi_{n-1,1}\boldsymbol{q}_1 - \xi_{n-1,2}\boldsymbol{q}_2) + d_{n4}\\
k_{n2} &= d_{n1}(\xi_{n1} - 1)q_{n1} \qquad (5)
\end{aligned}$$

where the e's and d's are coefficients yet to be determined. Note that p_{ni} is an affine function, k_{ni} is a product of q and ξ, and k_{ni} vanishes when the previous realized ξ equals its expected value (which is 1 in our case). Also note that k_{ni} appears linearly in the revenue function $R(q)$. As a result, the revenue will be quadratic. Thus, it is conceivable that by choosing p and k appropriately, the consumer's criterion $E[S-R]$ can be modified to agree with that of the producer, since both will be quadratic under the pricing strategy of (5). Here, the incentive pricing strategy simply fulfills the old adage, "If you wish other people to behave in your interest, then make them see things your way." The result is presented in Proposition 3. We shall first define several variables.

$$\begin{aligned}
y_{n1} &= (a_{n3} + a_{n2}b_{n3})d_{n3} + b_{n3}e_{n3} \qquad \text{for } 2 \leq n \leq N\\
&= 0 \qquad \text{for } n = N+1\\
y_{n2} &= y_{n1} + x_{n2} - c\\
y_{n3} &= (1 - a_{n2}^2)y_{n2} + (w_1 - w_2)\\
y_{n4} &= -[a_{n4} + a_{n2}b_{n4} - a_{n2}(1 - b_{n1})\boldsymbol{q}_1 + a_{n1}\boldsymbol{q}_2]d_{n3}\\
&\qquad - (b_{n4} + b_{n1}\boldsymbol{q}_1)e_{n3} + w_4[-a_{n4} - (1 + a_{n2})b_{n4}\\
&\qquad + (1 - a_{n1})\boldsymbol{q}_2 + (1 + a_{n2})(1 - b_{n1})\boldsymbol{q}_1] + e_{n1}\\
&\qquad\qquad \text{for } 2 \leq n \leq N\\
&= e_{N+1,1} \qquad \text{for } n = N+1\\
y_{n5} &= a_{n2}a_{n3}y_{n+1,2} - w_4\\
y_{n6} &= -\{a_{n+1,4} + a_{n+1,2}b_{n-1,4} - a_{n2}a_{n4}(a_{n+1,3}\\
&\qquad + a_{n+1,2}b_{n+1,3}) - a_{n+1,2}(1 - b_{n+1,1})\boldsymbol{q}_1\\
&\qquad + [a_{n-1,1} - (1 + a_{n1}a_{n2})\\
&\qquad \cdot (a_{n-1,3} + a_{n+1,2}b_{n-1,3})]\boldsymbol{q}_2\}d_{n+1,3}\\
&\qquad - [b_{n+1,4} - a_{n2}a_{n4}b_{n+1,3} + b_{n+1,1}\boldsymbol{q}_1\\
&\qquad - (1 + a_{n1}a_{n2})b_{n-1,3}\boldsymbol{q}_2]e_{n-1,3}\\
&\qquad - w_4(w_2 + c)[a_{n+1,4} + (1 + a_{n+1,2})b_{n+1,4}]\\
&\qquad + a_{n4}[(1 + a_{n2})(x_{n+1,2} - c) - w_2]\\
&\qquad + w_4(1 + a_{n+1,2})(1 - b_{n+1,1})(w_2 + c)\boldsymbol{q}_1\\
&\qquad + \{w_4(1 - a_{n-1,1})(w_2 + c)\\
&\qquad + a_{n1}[(1 + a_{n2})(x_{n+1,2} - c) - w_2]\\
&\qquad + c(x_{n+1,2} - w_2 - c)\}\boldsymbol{q}_2 + d_{n1}\\
&\qquad\qquad \text{for } 1 \leq n < N\\
&= (a_{N4} + a_{N1}\boldsymbol{q}_2)[(1 + a_{N2})(x_{N+1,2} - c) - w_2]\\
&\qquad + c(x_{N+1,2} - w_2 - c)\boldsymbol{q}_2 + d_{N1} \qquad \text{for } n = N.
\end{aligned}$$

($AS1$) $y_{n2} > 0$ and $y_{n3} > 0$.

Proposition 3: If ($AS1$) holds for $2 \leq n \leq N+1$, then by choosing

$$\begin{aligned}
d_{n1} &= (1 - b_{n1})y_{n+1,3}\boldsymbol{q}_1\\
d_{n2} &= -(1 + a_{n2})y_{n+1,2} + w_2\\
d_{n3} &= -a_{n3}y_{n+1,2} - w_4\\
d_{n4} &= y_{n+1,4} - a_{n4}y_{n+1,2}\\
e_{n1} &= (1 - a_{n-1,1})y_{n2}\boldsymbol{q}_2\\
e_{n3} &= y_{n5} - b_{n3}y_{n-1,3}\\
e_{n4} &= y_{n6} - b_{n4}y_{n+1,3}
\end{aligned}$$

the optimal consumption levels are given by

$$\begin{aligned} q_{n2}^* = &[(y_{n+1,2}q_2 - e_{n+1,1})\xi_{n2} \\ &-(y_{n+1,2} - w_2)(q_{n1} - \xi_{n1}q_1) \\ &- w_4(q_{n-1,1} + q_{n-1,2} - \xi_{n-1,1}q_1 \\ &- \xi_{n-1,2}q_2) + (y_{n+1,4} - p_{n2})]/y_{n+1,2} \\ q_{n1}^* = &[(y_{n-1,3}q_1 - d_{n1})\xi_{n1} \\ &+ y_{n5}(q_{n-1,1} + q_{n-1,2} - \xi_{n-1,1}q_1 \\ &- \xi_{n-1,2}q_2) + (y_{n6} - p_{n1})]/y_{n+1,3}. \end{aligned} \tag{6}$$

If p_{n1} and p_{n2} are substituted by (5) with coefficients given above, then $q_{n1}^* = q_{n1}^t$ and $q_{n2}^* = q_{n2}^t$.

Note that although the above formulas for y_{ni}, d_{nj}, and e_{nk} seem quite complicated, it is only important to know here that if $(AS1)$ is satisfied, then the consumer can be induced to behave cooperatively, and the coefficients of pricing strategies can be precalculated. $(AS1)$ is the second order condition that guarantees q^* to be the true maximal strategies. We shall see in the next section that as N approaches infinity, the sequences $\{y_{n2}\}$, $\{y_{n3}\}$ converge and $(AS1)$ is satisfied. Note also that $d_{n1} > 0$ (the coefficient of the fixed charge for period $n2$) which says that if the realized ξ_{n1} is greater than its expected value (which is 1), then the fixed charge for the next period (k_{n2}) will be positive. Similarly, $e_{n1} > 0$.

C. Comparison Between Peak Load Pricing and Load Adaptive Pricing

Let J^*_{lap} and J^t_{lap} be, respectively, the optimal payoffs for the Stackelberg game and its corresponding team problem under the load adaptive pricing formulation. We can also formulate a Stackelberg game and its corresponding team problem under the peak load pricing formulation having the information structure as shown in Fig. 1(b). Let J^*_{plp} and J^t_{plp} be, respectively, their optimal payoffs. It is well-known that for team problems, more information means equal or better payoffs. In our model it is easy to show by direct calculation that $J^t_{\text{plp}} < J^t_{\text{lap}}$. Thus, from Proposition 3, we have

$$J^*_{\text{plp}} \leq J^t_{\text{plp}} < J^t_{\text{lap}} = J^*_{\text{lap}}.$$

Thus, the model predicts that load adaptive pricing is desirable. Note that, however, the implementation costs of load adaptive pricing are not considered here. If the energy costs increase at a much faster rate than the implementation costs of hardware and software, it is reasonable to believe that the adoption of load adaptive pricing will be justified in the future.

V. The Case with Infinite Cycles

In the previous section, the load adaptive pricing problem was treated for the general N-cycle case. As mentioned earlier, a cycle may be a day, a week, etc. Once a pricing scheme is set, it may remain in effect for several months and even years. Thus, it is very important to examine the asymptotic behavior of the solution. Two questions will be addressed here. First, will the coefficients of the solutions converge? That is, will the consumer face a fixed set of coefficients for each cycle under consideration? If these coefficients do not converge, the consumer will face a different set of coefficients each cycle, which is very undesirable. The second question is whether or not the system is stable. That is, if some small disturbance occurred at the nth cycle, will it die out as time passes? If the system is not stable, any small disturbance occurring either in consumption or in pricing will cause future q, p, and k to oscillate forever. We shall investigate the team solution first.

A. Convergence of the Team Solution and Its Stability

The team solution is given by (3), where the coefficients are determined by Proposition 1. In the investigation of convergence of coefficients, x_{n1} plays a key role. It is easy to see that if the sequence $\{x_{n1}\}$ converges, then the rest of the sequences will converge. We have the following results.

Proposition 4: As N approaches infinity, $\{x_{n1}\}$ converges to x_1 where

$$\begin{aligned} x_1 &= \tfrac{1}{2}\left[-x_4 + (x_4^2 - 4)^{1/2}\right] \qquad \text{for } w_4 \neq 0 \\ &= 0 \qquad \text{for } w_4 = 0 \end{aligned}$$

and when $w_4 \neq 0$,

$$\begin{aligned} x_4 = &2 + w_3/w_4 \\ &+ (w_1 + c)(w_2 + c)/[w_4(w_1 + w_2 + c)]. \end{aligned}$$

As a result, $\{x_{n2}\}$, $\{x_{n3}\}$ and all the coefficients of the team solution $\{a_{ni}\}$, $\{b_{nj}\}$ converge (to x_2, x_3, a_i, and b_j, respectively). Furthermore,

$$\begin{aligned} &-1 < x_1 \leq 0 \\ &x_2 > (w_2 + w_3 + w_4 + c) > 0 \\ &x_3 > [(w_1 + c)(w_2 + c) \\ &\quad + (w_3 + w_4)(w_1 + w_2 + 2c)] > 0 \end{aligned} \tag{7}$$

and the team solution exists.

It is easy to derive from Proposition 4 that

$$\begin{aligned} &1 > a_1 > 0, \qquad a_2 < 0, \qquad a_3 \leq 0 \\ &1 > b_1 > 0, \qquad b_3 \leq 0 \end{aligned}$$

and the interpretation which was applied to the finite-cycle case still prevails.

We shall now investigate the stability property. Assume some disturbance occurred at the n'th cycle and the team strategies are still followed afterwards. From (3), disturbances propagate according to the following formulas:

$$\begin{aligned} \Delta q_{n1} &= b_3(\Delta q_{n-1,1} + \Delta q_{n-1,2}) \\ \Delta q_{n2} &= a_2 \Delta q_{n1} + a_3(\Delta q_{n-1,1} + \Delta q_{n-1,2}) \\ &= (a_3 + a_2 b_3)(\Delta q_{n-1,1} + \Delta q_{n-1,2}) \end{aligned} \tag{8}$$

and

$$(\Delta q_{n1} + \Delta q_{n2}) = x_1(\Delta q_{n-1,1} + \Delta q_{n-1,2}) \tag{9}$$

where Δq_{nm} is the deviation from the nominal consumption level. Since $-1 < x_1 \leq 0$ as stated in Proposition 4, the sum of deviations $\Delta q_{n1} + \Delta q_{n2}$ will die out. So will Δq_{n1} and Δq_{n2} [from (8)]. Thus, *the team solution is stable*. This is intuitively reasonable since optimality usually implies stability.

B. Convergence of the Pricing Scheme and Its Stability

The pricing scheme is given by (5) where the coefficients are determined by Proposition 3. In order to investigate the convergence of these coefficients, we shall assume that all a_{ni} and b_{nj} have converged. Similar to the case treated in the previous subsection, y_{nl} plays a key role here. If $\{y_{n1}\}$ converges, so do all other sequences. The following proposition states the results.

Proposition 5: As N approaches infinity, $0 < (a_3^2 - b_3^2 - a_2^2 b_3^2) < 1$, and $\{y_{n1}\}$ converges to

$$\begin{aligned} y_1 = &-\left[1 + \left(a_3^2 + b_3^2 - a_2^2 b_3^2\right)\right]^{-1} \\ &\cdot \left[\left(a_3^2 + b_3^2 - a_2^2 b_3^2\right)(x_2 - c)\right. \\ &\left. + w_4 x_1 + b_3^2 (w_1 - w_2)\right]. \end{aligned} \tag{10}$$

As a result, $\{y_{ni}\}$ and all the coefficients of the pricing rules $\{d_{nj}\}$ and $\{e_{nk}\}$ converge (to y_i, d_j, and e_k, respectively). $y_2 > 0$ and $y_3 > 0$, i.e., $(AS1)$ is satisfied; thus q_{nm}^*, p_{nm}, and k_{nm} exist and $q_{nm}^* = q_{nm}^t$.

Since the consumer can be induced to behave cooperatively ($q_{nm}^* = q_{nm}^t$), one would expect that the stability of the system follows from the stability of the team problem. This is indeed the case. More precisely, disturbances propagate according to the following formulas [from (5) and (6)]:

$$\begin{aligned} \Delta p_{n1} &= e_3(\Delta q_{n-1,1} + \Delta q_{n-1,2}) \\ \Delta p_{n2} &= d_2 \Delta q_{n1} + d_3(\Delta q_{n-1,1} + \Delta q_{n-1,2}) \\ \Delta q_{n1} &= \left[y_5(\Delta q_{n-1,1} + \Delta q_{n-1,2}) - \Delta p_{n1}\right]/y_3 \\ &= (y_5 - e_3)(\Delta q_{n-1,1} + \Delta q_{n-1,2})/y_3 \\ \Delta q_{n2} &= -\left[(y_2 - w_2)\Delta q_{n1}\right. \\ &\quad \left. + w_4(\Delta q_{n-1,1} + \Delta q_{n-1,2}) + \Delta p_{n2}\right]/y_2 \\ &= -\left[(y_2 - w_2 + d_2)\Delta q_{n1} + (w_4 + d_3)\right. \\ &\quad \left. \cdot (\Delta q_{n-1,1} + \Delta q_{n-1,2})\right]/y_2 \\ \Delta q_{n1} + \Delta q_{n2} &= \left[(w_2 - d_2)(y_5 - e_3)/y_3 - (w_4 + d_3)\right] \\ &\quad \cdot (\Delta q_{n-1,1} + \Delta q_{n-1,2})/y_2 \\ &= x_1(\Delta q_{n-1,1} + \Delta q_{n-1,2}) \end{aligned} \tag{11}$$

which is identical to (9). Thus, $\Delta q_{n1} + \Delta q_{n2}$ will die out; so will Δq_{n1}, Δq_{n2}, Δp_{n1}, and Δp_{n2}, and the system is stable.

Example 1: Consider an infinite-cycle load adaptive pricing model with the following parameters:

$$w_1 = w_2 = 1, \quad w_3 = 1/2,$$
$$w_4 = 1/4, \quad c = 2.$$

q_{nm}^t, p_{nm}, k_{nm}, and q_{nm}^* are given by (3), (5), and (6), respectively, where the coefficients are determined by Propositions 1, 3, 4, and 5 as follows:

$$\begin{array}{lll} a_1 = 0.497, & a_2 = -0.245, & a_3 = -0.063 \\ b_1 = 0.465, & b_3 = -0.051 & \\ y_2 = 1.987, & y_3 = 1.868, & y_5 = -0.219 \\ d_1 = \boldsymbol{q}_1, & d_2 = -0.500, & d_3 = -0.125 \\ e_1 = \boldsymbol{q}_2, & e_3 = -0.125. & \end{array}$$

The condition $a_1 > 0$, $a_2 < 0$, $a_3 < 0$, $b_1 > 0$, $b_3 < 0$, $d_1 > 0$, and $e_1 > 0$ can be interpreted as before. In this example we also have

$$d_2 < 0, \qquad d_3 < 0, \qquad e_3 < 0.$$

The negativity of d_2 indicates that if q_{n1} is small and the deviation from the ideal consumption in period $n1$, $(\xi_{n1}\boldsymbol{q}_1 - q_{n1})$ is large, then p_{n2} will be high. The utility company does this since it anticipates a high demand in period $n2$ caused by the large deviation in period $n1$. Similarly, if q_{n1} is very large and $(\xi_{n1}\boldsymbol{q}_1 - q_{n1})$ becomes negative, p_{n2} will be low. $d_3 < 0$ and $e_3 < 0$ are interpreted in the same way.

VI. Conclusions

The motivation for our research may be stated in Schweppe's words [26]: "Computing and communication are among the few things left in our society that are decreasing in cost. Furthermore, data-network communications and mini- and microcomputer technology are evolving at a rate that parallels the needs of electric power systems. Future control systems will exploit this technology extensively." And by the year 2000, "multilevel controls and home minis will enable utilities to buy and sell power at *real time* rates determined by supply and demand."

The research work reported here was initiated by peak load pricing problems of electric systems. From a game theoretic point of view, peak load pricing problems are Stackelberg games where the utility company (the leader) has only open-loop control and customers (followers) are not provided with any system demand/supply information. In recognizing the limit of the peak load pricing formulation and the persuasive breakthroughs in microelectronic technology, we formulate the load adaptive pricing problem mathematically where the utility company sells power at "real time" rates.

Load adaptive pricing is a closed-loop Stackelberg problem. In this paper, we have solved a load adaptive pricing problem for a particular producer/consumer model by using the methodology developed in [13] and [14]. We demonstrated that it is possible for the utility company to induce the customer to behave cooperatively to achieve the team optimum. As the number of cycles approaches infinity, our results show that the solution converges and the system is stable.

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An Optimal Control Approach to Dynamic Routing in Networks

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Abstract—**This paper explores the application of optimal control theory to the problem of dynamic routing in networks. The approach derives from a continuous state space model for dynamic routing and an associated linear optimal control problem with linear state and control variable inequality constraints. The conceptual form of an algorithm is presented for finding a feedback solution to the optimal control problem when the inputs are assumed to be constant in time. The algorithm employs a combination of necessary conditions, dynamic programming, and linear programming to construct a set of convex polyhedral cones which cover the admissible state space with optimal controls. An implementable form of the algorithm, along with a simple example, is presented for a special class of single destination networks.**

I. Introduction

THE MODEL considered in this paper is motivated by the following problem: given a capacitated network, an initial accumulation of traffic at the nodes, and possible input traffic and assuming that the accumulated traffic can be measured at all times at all nodes, clear the traffic congestion so that the total traffic delay is minimized. The problem can be applicable to the communication networks where messages may accumulate in node buffers, transportation networks where we have vehicle traffic [8], or other types of networks. In all cases, we address the problem of *dynamic routing* [1], whereby the decision on how to forward traffic through the network is based on measurement of the instantaneous queue lengths at the network nodes.

A model for the analysis of dynamic network routing has been proposed in [1], whereby it has been shown that the problem gives rise to a dynamic linear continuous state space equation. The criterion considered in [1] is the minimum weighted message delay throughout the network giving rise to a *linear optimal control problem with linear state and control variable inequality constraints and with linear integral cost functional.* The inputs are assumed to be deterministic functions of time, and a feedback solution is sought which drives all of the state variables to zero at the final time.

Little theoretical or computational attention has been paid to the class of control problems with state variable inequality constraints and the control appearing linearly in the dynamics and performance index. In this case, the control is of the bang-bang variety and the costates may be characterized by a high degree of nonuniqueness. In [2] the necessary conditions associated with this problem are examined when the control and state constraints are both scalars, and an interesting analogy is presented between the junction conditions associated with state boundary arcs

Manuscript received January 18, 1980; revised September 15, 1980 and May 18, 1981. Paper recommended by L. G. Shaw, Past Chairman of the Networks and Transportation Systems Committee.

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0018-9286/82/0400-0329$00.75