$$C=[0 \quad 0 \quad 0 \quad 1],$$

$$A=\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}.$$

This system is unobservable for *every* constant input, in fact, the intersection of all the unobservable spaces with respect to all constant inputs is nonempty, although the system is an observable bilinear system!

The proofs of [9] are essentially of the existence type and the genericity of the required class of inputs is proved in an infinite-dimensional space (C^{∞}). This makes it difficult to use these results to obtain an algorithm for generating the desired inputs. In contrast to this we have proved genericity essentially in a finite-dimensional space, since Theorems 2 and 3 guarantee that the set of piecewise constant inputs with $n_0^2 p$ points of discontinuity is sufficient. This allows the development of the random algorithm presented in this section. Also, our proofs are straightforward and algebraic in nature without any heavy mathematical machinery being used—this seems appropriate since we have considered a simple bilinear system.

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Information Structure, Stackelberg Games, and Incentive Controllability

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Abstract—**Recently there has been considerable activity in the area of deterministic closed-loop Stackelberg games. It turns out that these results are closely related to various incentive problems and pricing problems in economics. We propose in this paper a unified treatment of these problems from the viewpoint of the information structure of a general two-person nonzero-sum game. This treatment lays bare the underlying ideas and permits easy extensions to stochastic cases. Single stage, linear-quadratic-Gaussian Stackelberg problems are then examined in detail. Examples from electricity pricing and organizational design are also discussed as illustrations of this general approach.**

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I. Introduction

Recently there has been considerable activity in the area of "closed-loop Stackelberg" problems. One reason for this interest is that it was thought for a long time that closed-loop Stackelberg solutions, even for the linear-quadratic deterministic problem, were difficult to obtain [1]. Then independently, Basar and Selbuz [2], Papavassilopoulos and Cruz [3], [17], and Tolwinski [4] all came up with solutions to this problem. What is more surprising, solutions in [2], [4], and [17] achieve the absolute optimal payoff for the leader.[1] In other words, the leader was able to induce the follower to behave as if the follower was also optimizing the payoff of the leader, and thus achieve the cooperative optimum. The techniques in [2], [3], and [4] appear superficially rather different and somewhat specialized and magical. Partly, this is due to the multistage nature of the problem. The underlying idea actually is quite simple and can be explained in a unified manner which will permit further extensions.[2] Moreover, Stackelberg problem has its roots in economics and, as will be shown, can be applied to explain many interesting problems in the theory of price and monopoly [5] and organizational design [6]. The purpose of this paper is to discuss these issues.

II. The Stackelberg Solution

A. Basic Concept

The Stackelberg solution concept, which was first introduced in economics in the 1930's within the context of static economic competition [7], has entered the control and game literature through the works of Chen, Cruz, and Simaan [8], [9], [1]. The Stackelberg solution is mostly appropriate in nonzero-sum two-person games when one of the players (the leader L) has the ability to declare and impose his strategy before the other player (the follower F). Let Γ_L and Γ_F be the strategy spaces for L and F, respectively, with $\gamma_L \in \Gamma_L$, $\gamma_F \in \Gamma_F$. Let $J_L(\gamma_L, \gamma_F)$ and $J_F(\gamma_L, \gamma_F)$ be their corresponding payoff functions as stated in the strategic form of the game. The basic idea is as follows. For any given choice of strategy γ_L made by L, F tries to choose a reaction strategy $\gamma_F^0 \in \Gamma_F$ which maximizes his own payoff J_F. L knowing F's rationale, wishes to announce a strategy γ_L^s such that with this strategy and F's reaction to it, L's maximum payoff is achieved. Mathematically, let

$$R(\gamma_L)=\left[\gamma_F^0 \in \Gamma_F : J_F(\gamma_L, \gamma_F^0) \geqslant J_F(\gamma_L, \gamma_F) \text{ for all } \gamma_F \in \Gamma_F\right] \quad (2.1)$$

be the rational reaction set of F. Then L chooses γ_L^s such that

$$\inf J_L(\gamma_L^s, \gamma_F) \geqslant \inf J_L(\gamma_L, \gamma_F) \qquad \text{for all } \gamma_L \in \Gamma_L \quad (2.2)$$

where the infimum is taken over $\gamma_F \in R(\gamma_L^s)$ on the left-hand side of the inequality and over $\gamma_F \in R(\gamma_L)$ on the right-hand side of the inequality. Thus, unlike the Nash equilibrium solution concept, the roles of the players are not symmetric in the Stackelberg case.

It is important to note that strategies of both players are required to be declared prior to the beginning of the game, with L having the right to announce first. However, the sequence of actions during the course of the game involves the rules of the game in the extensive form, and F may be required to act before L. *It is the order of announcing strategies rather than the order of actions that distinguishes L from F.*

[1] In [3] a set of necessary conditions for Stackelberg strategies of a deterministic, continuous dynamic system was derived. Whether the absolute optimal payoff could be achieved was discussed in Section 4 of [3]. Reference [17] came to our attention after the writing of this paper.

[2] After writing the earlier version (November 13, 1979) of this paper, we have discovered that in a paper to be published [16] Basar also extended the problem to the stochastic case. See more discussion later in Sections III and IV.

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B. The Role of Information

The *information structure* for a game characterizes the precise information gained or recalled by each player at every stage of the game. Precise delineation of information structures is very important because different information structures characterize different rules, lead to different interpretations, and also yield different results.

The class of Stackelberg problems studied in [2], [4], and this paper can be interpreted (in a sense that is loosely stated now, but will be made precise later) as having the following kind of information structure: F is required to act first, L acts *after* F, knowing the actual choice of F. In other words, L's strategy can be a function of F's decision. That is, while L announces his strategy first, F actually acts first. For lack of a better name, we call such problems Stackelberg problems with *reversed information structure* or simply *reversed Stackelberg problems*.

Several important conceptual points will be noted with respect to the reversed information structure. Assume decisions of L affect the payoff of F (if not, then there will not be a Stackelberg problem). There are two reasons why L may be interested in the decision of F.

R1): F may know something (about the states of nature) that interests L. Knowing F's strategy and what he has done may enable L to infer the states of nature. This occurs in stochastic problems where F's information structure is not nested in that of L.

R2): F's decision may directly affect the payoff of L. This occurs in deterministic problems, as well as stochastic problems with both nested and nonnested information structures.

The trick here is somehow to induce F to act in such a way that benefits L directly as in R2) and/or indirectly as in R1). The mechanism to do this is for L to include the proper inducement (incentives or threats) while announcing his strategy. It is not too surprising that in many cases this can be accomplished if L's strategy can be a function of F's action.

A simpleminded approach would be to declare that unless F chooses such and such action, L will severely punish F via L's decision, which by assumption can affect F's payoff. This is in fact the basic idea. However, there are a couple of minor problems with this. First, the severe penalty may also do damage to L's own payoff. Thus, such a threat may not be believable. We would like to avoid unbelievable threats. Second, in the case of R1), L cannot simply force F to disclose his knowledge of the states of nature since there is no way for L to verify its truth directly. Otherwise, L would not need F to tell him what he knows about the states of nature. What L can do, however, is to induce F to act in such a way as to reveal the truth. For example, in a public project the government cannot simply declare that the cost of the project will be divided in proportion to the benefit that the project accrues to each participant, since the government cannot easily verify the truth of what each participant may declare as the benefit. The government can, however, choose to declare how the project will be carried out based solely on what each participant chooses to reveal as the benefit so as to induce each participant to reveal the truth in his own interest. Thus the crucial ingredient here is the specialized information structure; among other things, L's strategy is a function of F's decision. The key problem is the design of L's strategy.

In the work of Basar, Papavassilopoulos, and Tolwinski only R2) is operative since the problem is either deterministic or has a nested information structure. However, R1) becomes important in economic problems of the type mentioned above. Of course, in general both R1) and R2) may be operative. In the next section we shall first formalize these notions in terms of a single-stage problem.

III. Single-Stage Stackelberg Problems (SSP)

A. Classification of Single-Stage Stackelberg Problems

Let $\xi=(\xi_1, \xi_2,\cdots,\xi_m)^T$ be a vector of random variables with given distribution. ξ represents all the uncertainties that have a bearing on the problem and is often called "the states of nature." Let z_L be the information available to L, and γ_L be a measurable mapping from the information space Z_L generated by z_L to the decision space U_L of dimension r_L. Let the payoff function for L be $J_L=E[L_L(\gamma_L,\ \gamma_F,\ \xi)]$. Corresponding quantities for F are defined similarly. SSP can be classified into three categories according to the sequence of actions by L and F.

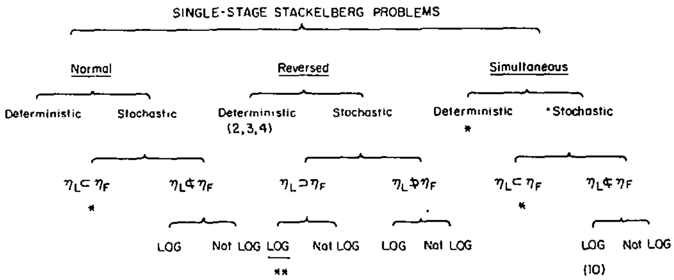

Fig. 1. Single-stage Stackelberg problems. * = The follower has a permanently optimal solution. ** = The class of problems discussed in Section III-B.

1) Normal Version: L announces γ_L and also acts first. The information structure is

$$\eta_L:\ Z_L(\xi)$$
$$\eta_F:\ Z_F(u_L,\xi)$$

where

$$u_L=\gamma_L(z_L),\quad z_L=Z_L(\xi)\quad\text{and}\quad z_F=Z_F(u_L,\xi).$$

2) Reversed Version (RSSP): L announces γ_L first, but acts after F. We have

$$\eta_L:\ Z_L(u_F,\xi)$$
$$\eta_F:\ Z_F(\xi)$$

where

$$u_F=\gamma_F(z_F),\quad z_L=Z_L(u_F,\xi)\quad\text{and}\quad z_F=Z_F(\xi).$$

3) Simultaneous Version: L announces γ_L first and the players act simultaneously in the sense that neither player knows the action of the other player, The information structure is

$$\eta_L:\ Z_L(\xi)$$
$$\eta_F:\ Z_F(\xi)$$

where

$$z_L=Z_L(\xi),\quad z_F=Z_F(\xi).$$

We shall assume that observations on u_L, u_F are noiseless. Thus, for the normal version we can separate u_L from Z_F and write the information available to F as

$$\eta_F:\ u_L,Y_F(\xi)$$

where $y_F=Y_F(\xi)$ is independent of u_L. Similarly, for RSSP we have

$$\eta_L:\ u_F,Y_L(\xi)$$

where $y_L=Y_L(\xi)$ is independent of u_F.

Each category of SSP can be further classified into different cases, as shown in Fig. 1. The cases which have been already treated in the literature, are indicated by appropriate references. Our concern in this paper is mainly the RSSP.

B. A Method to Solve RSSP With a Nested Information Structure

RSSP with nested information structure is considered in this section. The case where the information is not nested and R1) is operative will be discussed in Section VI. The essence of the method is to announce L's strategy in such a way as to induce F to choose the optimal team strategy. Assuming full cooperation, L's payoff function is first maximized over the strategy spaces of L *and* F, yielding optimal team strategies for both players denoted as γ_L^t and γ_F^t. Now we consider the following strategy for L

$$\gamma_L^*=f\big(u_F,y_L,\gamma_F^t(z_F),\gamma_L^t(z_L)\big)\tag{3.1}$$

where f has the property that when $u_F=\gamma_F^t(z_F)$, then $f=\gamma_L^t(z_L)$ for all z_F and z_L. For example, $\gamma_L^s=A(u_F-\gamma_F^t)+\gamma_L^t$, where A is some constant matrix. For each declaration of f and hence γ_L^s, F responds according to $\gamma_F^0(z_F;\gamma_L^s)\in R(\gamma_L^s)$. We wish to choose γ_L^s such that

$$\gamma_F^s(z_F)=\gamma_F^0(z_F;\gamma_L^s)=\gamma_F^t(z_F) \qquad \text{for each } \gamma_F^0\in R(\gamma_L^s). \quad (3.2)$$

If this is possible, then we have achieved the absolute optimum for L's payoff which obviously upper bounds the solution of the RSSP. Consequently, we have solved the RSSP.[3]

The above discussion will now be elaborated mathematically in terms of steps S1)–S3).

S1): Determine Team Strategy—Let us consider the corresponding team problem, where the two players work together to maximize J_L under the *original* information structure, that is, the problem

$$\max_{\gamma_L,\gamma_F} E[L_L(\gamma_L,\gamma_F,\xi)].$$

We shall assume:

A1): The team problem has a solution (γ_L^t,γ_F^t).

S2): Determine F's Reaction—For RSSP, given any $\gamma_L(u_F,y_L)$, F wishes to maximize J_F

$$\max_{\gamma_F} E[L_F(\gamma_L,\gamma_F,\xi)]$$

or, in extensive form

$$\max_{u_F} E/z_F[L_F(\gamma_L,u_F,\xi)].$$

This optimization process yields

$$\gamma_F^0(z_F;\gamma_L)\in R(\gamma_L). \quad (3.3)$$

S3): Determine L's Stackelberg Strategy—Taking (3.3) into account, L now has to maximize J_L

$$\max_{\gamma_L} E\Big[L_L\big(\gamma_L,\gamma_F^0(z_F;\gamma_L),\xi\big)\Big]. \quad (3.4)$$

If we can find $\gamma_L^s(u_F,y_L)$ such that

$$\gamma_F^0(z_F;\gamma_L^s)=\gamma_F^t(z_F) \quad (3.2')$$

and

$$\gamma_L^s\big(\gamma_F^t(z_F),y_L\big)=\gamma_L^t\big(\gamma_F^t(z_F),y_L\big) \quad (3.1')$$

then (γ_L^s,γ_F^t) achieves the team optimal J_L^t, and hence solves the RSSP.

Thus, the problem is reduced to finding γ_L^s as in step S3). In [2], [4], and [11] γ_L^s has the form

$$\gamma_L^s(u_F,y_L)=g\big(\gamma_F^t,u_F,y_L\big)+\gamma_L^t(u_F,y_L) \quad (3.5)$$

where g is some function having the property that when $u_F=\gamma_F^t$, $g\equiv 0$. Note that (3.5) is implementable if the following condition holds.

A2): The information structure is nested, i.e., $Z_F c Z_L$. Also, note that when γ_L^s has the form in (3.5), the satisfaction of (3.2′) guarantees the same for (3.1′). For our problem let us consider

$$\gamma_L^s(u_F,y_L)=A(y_L)\big[u_F-\gamma_F^t(z_F)\big]+\gamma_L^t\big(\gamma_F^t(z_F),y_L\big). \quad (3.6)$$

If an $r_L\times r_F$ matrix $A(y_L)$ can be determined such that γ_L^s of (3.6) satisfies (3.2′), then our problem is solved. If such an $A(y_L)$ exists, we say that the RSSP is *linearly incentive-controllable* ($l.i.c.$).

[3]It has been shown in [2], [4], and [11] that within the context of a specific class of *deterministic* dynamic problems, a similar approach can effectively be used to obtain closed-loop Stackelberg strategies. Here, we extend their approach to two-person discrete-time *stochastic* setup, but limited to RSSP. As mentioned before, a similar stochastic setup has been considered in [16]. However, a special form of f is used ($f=\gamma_L^t+S(u_F-\gamma_F^t)$, where S is a *constant matrix*), which restricts the solution and also leads to misleading claims. See Case 1 and Proposition 1 in Section IV.

IV. Linear-Quadratic-Gaussian Stackelberg Problem

We shall illustrate linear incentive controllability for the LQG Stackelberg problem where:

A3): J_L, J_F are quadratic in the decision variables of L and F and in the nature's decision (u_L, u_F, and ξ, respectively); ξ is Gaussian; y_F and y_L are linear in ξ.

More specifically,

$$J_L=E[L_L]=E\Big[\frac{1}{2}u_L'D_{11}u_L+u_L^TD_{12}u_F+\frac{1}{2}u_F'D_{13}u_F+u_L^TC_{11}x +u_F^TC_{12}x+u_L^Tk_{11}+u_F^Tk_{12}\Big]$$

$$J_F=E[L_F]=E\Big[\frac{1}{2}u_F'D_{22}u_F+u_F^TD_{21}u_L+\frac{1}{2}u_L'D_{23}u_L+u_L^TC_{21}x +u_F^TC_{22}x+u_L^Tk_{21}+u_F^Tk_{22}\Big]$$

where $x\sim N(0,\Sigma)$, with $\dim(x)=n$. D_{ij}, C_{ik}, $i,k=1,2$, $j=1,2,3$, are constant matrices of appropriate dimensions, $D_{ii}<0$; k_{ij} are appropriate dimensional constant vectors. The observations are

$$y_i=H_ix+w_i, \qquad i=1,2$$

where $\dim(y_i)=m_i$, $w_i\sim N(0,\Lambda_i)$, and H_i are appropriate dimensional constant matrices for $i=1,2$. x, w_1, and w_2 are independent random variables with $\Sigma>0$ and $\Lambda_i\geqslant 0$. Note that x, w_1, and w_2 now constitute the state of nature (ξ). The information structure is

$$\eta_L: u_F, Y_1, Y_2$$
$$\eta_F: Y_2.$$

We shall now solve the LQG-RSSP following the steps S1)–S3).

S1): Determine Team Strategy—In view of the nested LQG assumption [A2), A3)], the solution to the team problem [which exists by A1)][4] must be such that γ_L^t and γ_F^t are affine in z_L and z_F [12]. This solution, obtained using standard techniques, is

$$\gamma_L^t=-D_{11}^{-1}D_{12}u_F-D_{11}^{-1}C_{11}E[x|y_1,y_2]-D_{11}^{-1}k_{11} \quad (4.1)$$

$$\gamma_F^t=-\big(D_{13}-D_{12}^TD_{11}^{-1}D_{12}\big)^{-1}\Big[\big(C_{12}-D_{12}^TD_{11}^{-1}C_{11}\big)E[x|y_2] -D_{12}^TD_{11}^{-1}k_{11}+k_{12}\Big]. \quad (4.2)$$

Note that

$$E[x|y_2]=\Sigma_2y_2$$
$$E[y_1|y_2]=H_1\Sigma_2y_2$$
$$E[x|y_1,y_2]=Ky_1+(I-KH_1)\Sigma_2y_2$$

where

$$\Sigma_i=\Sigma H_i^T\big(H_i\Sigma H_i^T+\Lambda_i\big)^{-1}$$
$$K=PH_1^T\big(H_1PH_1^T+\Sigma_1\big)^{-1}$$
$$P=\Sigma-\Sigma H_2^T\Sigma_2^T$$

assuming, as we do here, that the inverse matrices are well defined. For simplicity, we shall just write

$$\gamma_L^t=h_{11}y_1+h_{12}y_2+h_{13}u_F+h_{10}$$
$$\gamma_F^t=h_{22}y_2+h_{20}$$

where h_{ij}, $i=1,2$, $j=0,1,2,3$, are constant matrices of appropriate dimensions. For ease of discussion, we shall make the following working assumption.

A4): y_1 and y_2 are independent. Note that if A4) does not hold we can always subtract from y_1 the orthogonal projection of y_1 on y_2 ($H_1\Sigma_2y_2$) to

[4]In the LQG context we shall further assume $D_{11}<0$, $D_{13}-D_{12}^TD_{11}^{-1}D_{22}<0$.

obtain $\hat{y}_1$, which can be thought of as the "innovation" part of y_1. The new, but equivalent, $\hat{y}_1$, y_2 then satisfies A4).

S2): Determine F's Reaction—We rewrite (3.6) as

$$\gamma_L^s = A(y_1, y_2)(u_F - h_{22}y_2 - h_{20}) + h_{11}y_1 + h_{12}y_2 + h_{13}u_F + h_{10} \tag{4.3}$$

and then substitute it into J_F and differentiate the resulting function with respect to u_F. Thus, we obtain the necessary condition for γ_F^s

$$\begin{aligned} E/y_2\Big[& D_{22}u_F + D_{21}\big(A(u_F - h_{22}y_2 - h_{20}) + h_{11}y_1 + h_{12}y_2 + h_{13}u_F \\ & + h_{10}\big) + \big(h_{13}^T + A^T\big)D_{21}^T u_F + \big(h_{13}^T + A^T\big)D_{23}\big(A(u_F - h_{22}y_2 \\ & - h_{20}) + h_{11}y_1 + h_{12}y_2 + h_{13}u_F + h_{10}\big) + \big(h_{13}^T + A^T\big)(C_{21}x + k_{21}) \\ & + C_{22}x + k_{22}\Big] = 0. \end{aligned} \tag{4.4}$$

To enable us to solve for u_F explicitly, we shall further assume that

A5):

$$E/y_2\Big[D_{22} + D_{21}(h_{13} + A) + \big(h_{13}^T + A^T\big)D_{21}^T + \big(h_{13}^T + A^T\big)D_{23}(h_{13} + A)\Big] < 0$$

for all possible y_2. Note that A5) is guaranteed if $D_{21} = 0$, i.e., the effect of u_L on J_F is additive. Also, if A is not a function of y_2, then the left-hand side of the above equation is a constant [in view of A4)] and A5) will be easy to check.

S3): Determine L's Stackelberg Strategy—We now seek to determine the conditions under which (3.2′) is satisfied. We shall first consider $A(y_1, y_2)$ of the following additive form:

$$A(y_1, y_2) = A_0 + A_1(y_1) + A_2(y_2) \tag{4.5}$$

where A_i is a function of y_i only for $i = 1$, 2, and A_0 contains all the constant terms. With A as in (4.5) and $u_F = \gamma_F^s$, (4.4) then becomes

$$\begin{aligned} & \Big[\big(D_{22} + D_{21}h_{13} + h_{13}^T D_{21}^T + h_{13}^T D_{23}h_{13}\big)h_{20} + D_{21}h_{10} \\ & \quad + h_{13}^T D_{23}h_{10} + h_{13}^T k_{21} + k_{22}\Big] + \big(A_0^T + E(A_1^T)\big)\big(D_{21}^T h_{20} \\ & \quad + D_{23}h_{13}h_{20} + D_{23}h_{10} + k_{21}\big) + E\big[A_1^T D_{23}h_{11}y_1\big] + E\big[A_1^T C_{21}Ky_1\big] \\ & \quad + \Big[\big(D_{22} + D_{21}h_{13} + h_{13}^T D_{21}^T + h_{13}^T D_{23}h_{13}\big)h_{22} + D_{21}h_{12} \\ & \quad + h_{13}^T D_{23}h_{12} + \big(h_{13}^T C_{21} + C_{22}\big)\Sigma_2 + \big(A_0^T + E(A_1^T)\big)\big(D_{21}^T h_{22} \\ & \quad + D_{23}h_{13}h_{22} + D_{23}h_{12} + C_{21}\Sigma_2\big)\Big] y_2 \\ & \quad + A_2^T\big(D_{21}^T h_{20} + D_{23}h_{13}h_{20} + D_{23}h_{10} + k_{21}\big) \\ & \quad + A_2^T\big(D_{21}^T h_{22} + D_{23}h_{13}h_{22} + D_{23}h_{12} + C_{21}\Sigma_2\big)y_2 = 0. \end{aligned} \tag{4.6}$$

Now in order for (4.6) to be an identity for all possible y_2, we need

$$A_2^T\big(D_{21}^T h_{22} + D_{23}h_{13}h_{22} + D_{23}h_{12} + C_{21}\Sigma_2\big)y_2 = 0.$$

A sufficient condition for this would be either

A6): $A_2(y_2) = 0$

or

A6′): $D_{21} = 0$, $D_{23} = 0$, $C_{21} = 0$.

The assumptions A6) and A6′) deserve further comment. One reason that A6) is required is this: The strategy (4.3) when substituted into J_F yields the nonlinear term $A_2(y_2)y_2$ which can not be equated with the linear term y_2 in γ_F^t. Similarly, requiring A6′) means that u_L only enters J_F linearly. Consequently, by choosing $A_2(y_2)$ to be linear, one can hope to realize (3.2′) or γ_F^t. In view of the above discussion, several cases of $A(y_1, y_2)$ are distinguished.

Case 1: $A = A_0$, a constant.

Case 2: $A = A_0 + A_1(y_1)$.

Case 3: $A = A_0 + A_2(y_2)$.

For each case, a sufficient condition for linear incentive-controllability is obtained. Essentially, it requires the existence of the matrix A satisfying (4.6) for all possible y_2. The results are summarized in three propositions after stating further assumptions. The messy details of the proofs are omitted since they are straightforward.

Case 1: $A = A_0$.

A7) There exists at least one constant matrix A_0 that satisfies the following equations:

$$B_1 A_0 = B_2 \tag{4.7}$$

$$B_3 A_0 = B_4 \tag{4.8}$$

where

$$B_1 = \big(D_{21}^T h_{20} + D_{23}h_{13}h_{20} + D_{23}h_{10} + k_{21}\big)^T$$

$$B_2 = -\Big[\big(D_{22} + D_{21}h_{13} + h_{13}^T D_{21}^T + h_{13}^T D_{23}h_{13}\big)h_{20} + D_{21}h_{10} + h_{13}^T D_{23}h_{10} + h_{13}^T k_{21} + k_{22}\Big]^T$$

$$B_3 = \big(D_{21}^T h_{22} + D_{23}h_{13}h_{22} + D_{23}h_{12} + C_{21}\Sigma_2\big)^T$$

$$B_4 = -\Big[\big(D_{22} + D_{21}h_{13} + h_{13}^T D_{21}^T + h_{13}^T D_{23}h_{13}\big)h_{22} + D_{21}h_{12} + h_{13}^T D_{23}h_{12} + \big(h_{13}^T C_{21} + C_{22}\big)\Sigma_2\Big]^T.$$ [5]

Proposition 1: If A1), A4), A5), and A7) hold, then the LQG–RSSP is l.i.c., γ_L^s is given by (4.5), where $A = A_0$ is obtained from (4.7) and (4.8). $\gamma_F^s = \gamma_F^t$.

This proposition is the same result as Theorem 3.1 of [16]. Note that the dimensions of A_0, B_1, B_2, B_3, and B_4 are $r_L \times r_F$, $l \times r_L$, $l \times r_F$, $n \times r_L$, $n \times r_F$, respectively. We have $r_L r_F$ variables to choose to satisfy $(n+1)r_F$ equations. Since the satisfaction of A7) is not generally guaranteed when $n > r_L - 1$, one might be tempted to conclude that γ_F^t can be induced only by making the penalty infinite. Cases 2 and 3 show otherwise.

Case 2: $A = A_0 + A_1(y_1)$.

A8): There exists at least one matrix $A(y_1) = A_0 + A_1(y_1)$ that satisfies the following equations:

$$B_5\big(A_0 + E(A_1)\big) = B_6 \tag{4.9}$$

$$E\big[A_1^T\big(-D_{23}D_{11}^{-1}C_{11} + C_{21}\big)Ky_1\big] = B_7 \tag{4.10}$$

where

$$B_5 = \big(D_{21}^T h_{22} + D_{23}h_{13}h_{22} + D_{23}h_{12} + C_{21}\Sigma_2\big)^T$$

$$B_6 = -\Big[\big(D_{22} + D_{21}h_{13} + h_{13}^T D_{21}^T + h_{13}^T D_{23}h_{13}\big)h_{22} + D_{21}h_{12} + h_{13}^T D_{23}h_{12} + \big(h_{13}^T C_{21} + C_{22}\big)\Sigma_2\Big]^T$$

$$\begin{aligned} B_7 = -\Big[& \big(D_{22} + D_{21}h_{13} + h_{13}^T d_{21}^T + h_{13}^T D_{23}h_{13}\big)h_{20} + D_{21}h_{10} + h_{13}^T D_{23}h_{10} \\ & + h_{13}^T k_{21} + k_{22} + \big(A_0^T + E(A_1^T)\big)\big(D_{21}^T h_{20} + D_{23}h_{13}h_{20} + D_{23}h_{10} + k_{21}\big)\Big]. \end{aligned}$$

Proposition 2: If A1), A4), A5), and A8) hold, then the LQG–RSSP is l.i.c., γ_L^s is given by (4.5), where $A = A_0 + A_1(y_1)$ is obtained from (4.9) and (4.10). $\gamma_F^s = \gamma_F^t$.

In this case the dimensions of B_5, B_6, and B_7 are $m_2 \times r_L$, $m_2 \times r_F$, and $r_F \times 1$, respectively. Equation (4.7) contains $m_2 r_F$ equations with $r_L r_F$ degrees of freedom. Thus, in general it is required that $r_L \geqslant m_2$.[6] Equation (4.10) contains r_F equations. If we let each component of A_1 to be a linear

[5] Note that h_{ij} are related to system parameters via (4.1) and (4.2).

[6] This is a reasonable requirement: after all, we are attempting to use u_L (dim r_L) to induce γ_F^t, which depends on y_2 (dim m_2).

function of y_1, then we can, in general, choose the $r_L r_F$ variables to satisfy (4.10).

Case 3: $A=A_0+A_2(y_2)$ and A6').

A9): There exists at least one matrix

$$A(y_2)=A_0+A_2(y_2)=A_0+\sum_{i=1}^{m_2} y_{2i}A_{2i} \tag{4.11}$$

where y_{2i} is the ith component of y_2 that satisfies the following equations for all possible y_2:

$$k_{21}^T A_0=-\left(D_{22}h_{20}+h_{13}^T k_{21}+k_{22}\right)^T \tag{4.12}$$

$$\sum_{i=1}^{m_2} y_{2i}A_{2i}^T k_{21}=-\left(D_{22}h_{22}+C_{22}\Sigma_2\right)y_2. \tag{4.13}$$

Proposition 3: If A1), A4), A5), A6'), and A9) hold, then the LQG-RSSP is l.i.c.. γ_L^s is given by (4.5) and (4.11), where $A(y_2)$ is obtained from (4.12) and (4.13). $\gamma_F^s=\gamma_F^t$.

Note that in (4.12) there are $r_L r_F$ variables to choose with r_F equations to be satisfied. In (4.13) there are $m_2 r_L r_F$ variables to choose with $r_2 m_2$ equations to be satisfied. Thus, solutions will, in general, exist.

We can also derive sufficient conditions for l.i.c. with $A(y_1,y_2)=A_0+A_1(y_1)+A_2(y_2)$ under A6') when neither A_1 nor A_2 is identically equal to zero. It turns out that the result is very similar to Proposition 3. Also, by allowing cross-product terms of y_1, y_2, in A, instead of the additive form (4.5), the above results can be extended somewhat. However, the underlying principle of solving the problem is still the same. We shall just illustrate this in the scalar case by adding a cross-product term $A_3 y_1 y_2$ to (4.5). This adds only one extra term to the left-hand side of (4.6)

$$A_3(D_{23}h_{11}+C_{21}K)E\left[y_1^2\right]y_2$$

which is linear in y_2. All other terms remain unchanged. Sufficient conditions can then be derived in the same manner as we have done.

The above development shows that there is considerable flexibility in these problems, and solutions are generally not unique. We shall illustrate further by means of examples.

Example 1: Consider an RSSP problem with LQG structure, where

$$J_L=E[L_L]=E\left[-\frac{1}{2}u_L^2+u_L u_F-u_F^2+x_1 u_L+x_2 u_F\right]$$

$$J_F=E[L_F]=E\left[-2u_F^2+(x_1+x_2)u_F+bu_L-u_F\right]$$

and x_1 and x_2 are independent zero-mean Gaussian random variables. Let the information structure be

$$\eta_L: u_F, x_1, x_2$$
$$\eta_F: x_2.$$

From (4.1) and (4.2) the team solution is

$$\gamma_L^t=u_F+x_1$$

$$\gamma_F^t=x_2.$$

Since A6') is satisfied, we shall use Proposition 3. Plugging appropriate numbers, (4.12), (4.13) become

$$bA_0=1-b$$

$$bA_2=3.$$

Thus, we can solve for the coefficients A_2 and A_0 if $b\neq 0$. In this case we have $A_2=3/b$ and $A_0=(1-b)/b$, and hence the RSSP is linearly incentive-controllable. When $b=0$, L's strategy does not affect the payoff function of F, and the RSSP is not linearly incentive-controllable.

Example 2: This is the same as Example 1, except that J_F is modified as follows:

$$J_F=E[L_F]=E\left[-\frac{1}{2}u_L^2-2u_F^2+2u_L x_2+(x_1+x_2)u_F+bu_L-u_F\right].$$

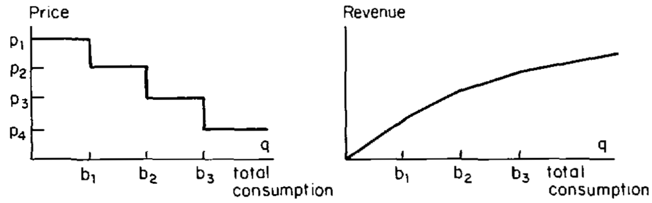

Fig. 2. Declining block rate.

Since A6') is not satisfied, we shall use Proposition 2. Equations (4.9) and (4.10) now become

$$A_0+E[A_1]=2$$
$$E[A_1 y_1]=3b-1.$$

We can let $A_1(y_1)=\hat{A}_1 y_1$, where $\hat{A}_1$ is a constant, then

$$A_0=2$$

$$\hat{A}_1=(3b-1)/E\left[x_1^2\right].$$

Thus,

$$\gamma_L^s=\left(2+(3b-1)y_1/E\left[x_1^2\right]\right)(u_F-x_2)-u_F-x_1$$

$$\gamma_2^s=\gamma_2^t$$

and the problem is l.i.c..

V. Declining Block Rate Via the Reversed Stackelberg Problem

In the electricity pricing context, the utility company (the producer) plays the role of the leader, and the customer (the consumer) plays the role of the follower.[7] For a given pricing strategy, the consumer determines his consumption strategy. The producer foresees this reaction and decides the optimal pricing strategy.

In general, the price of electricity may depend on the time of consumption and/or the level of consumption. *Declining block rate* is an example where the price depends on the level of consumption rather than on the time of consumption. The price is the highest for the first block of units consumed, after which it declines in a series of steps to lower and lower levels. As shown in Fig. 2, the dollar revenue $r(q)$ is a monotonic increasing piecewise linear function of q, the total kwh consumption of electricity. Let Γ_L be the set of all such functions.

In this section we shall show that the declining block rate problem (or the more general nonlinear pricing problem [5]) can be formulated as a *reversed* single-stage Stackelberg game. We shall assume a simple, but meaningful deterministic model.

$$J_F=\frac{1}{2}S\left[\bar{q}^2-(q-\bar{q})^2\right]-r(q)$$

$$J_L=r(q)-\frac{1}{2}cq^2$$

$$\text{Capacity constraint:}\quad q\leq\hat{q} \tag{5.1}$$

$$\text{Regulation:}\quad J_L\leq kq \tag{5.2}$$

where S, $\bar{q}$, c, $\hat{q}$, and k are some positive constants. The first term in J_F represents the degree of satisfaction that the consumer achieves by consuming at the level q, and the second term $r(q)$ is the cost of electricity to the consumer. It can be shown that J_F defined here is the conventional "Consumer's Surplus" [13], [14]. For the producer, $1/2cq^2$ represents the generating cost and J_L is the profit. Thus, the utility company in this model is a regulated profit maximizing monopoly with capacity limited to $\hat{q}$. For any given $r(\cdot)\in\Gamma_L$, the consumer chooses his consumption level q. The actual cost to the consumer will depend on q. The producer foresees this in deciding the optimal $r(\cdot)$. Thus, the problem is a RSSP with the following information structure:

[7] In the initial analysis we aggregate all customers into a single customer class. In later research we shall consider multiple customer classes.

η_L: q

η_F: $\varnothing$, no information.

We shall now solve this deterministic RSSP problem by the method described in Section III.

S1): Determine Team Strategy

$$\max_{r(\cdot),q} r(q)-\frac{1}{2}cq^2 \quad \text{subject to (5.1) and (5.2).}$$

The solution is

$$q^t=\hat{q}$$

and any $r(\cdot)\in\Gamma_L$ such that $r^t(\hat{q})=k\hat{q}+1/2c\hat{q}^2$.

S2): Determine F's Reaction—For any given $r(\cdot)\in\Gamma_L$, the reaction of F is

$$\max_q \frac{S}{2}\left[\bar{q}^2-(q-\bar{q})^2\right]-r(q).$$

The necessary condition is

$$Sq=S\bar{q}-\dot{r}(q) \tag{5.3}$$

where $\dot{r}(q)=dr(q)/dq$.

S3): Determine L's Stackelberg Strategy—Now we want to find $r^s(\cdot)$ such that $r^s(\hat{q})=r^t(\hat{q})$ and $q^s=\hat{q}$. Consider $r^s(\cdot)\in\Gamma_L$, with

$$r^s(\hat{q})=k\hat{q}+\frac{1}{2}c\hat{q}^2 \tag{5.4}$$

and

$$\dot{r}^s(q^s)=p$$

where $\dot{r}^s(q^s)$ is the value of $dr^s(q)/dq$ evaluated at q^s. From (5.3) we have

$$q^s=(S\bar{q}-p)/S.$$

To ensure $q^s=\hat{q}$, we need

$$p=S(\bar{q}-\hat{q}). \tag{5.5}$$

For $\dot{r}\geqslant 0$, it is necessary that

A10): $\bar{q}\geqslant\hat{q}$.

If A10) holds, then any $r(\cdot)\in\Gamma_L$, such that (5.4) and (5.5) are satisfied, is a candidate solution for the producer. However, since J_F is not concave (J_F is piecewise concave), (5.3) is not a sufficient condition for the global optimum of J_F. We need to check and make sure that the q derived from (5.3) is actually the global maximum of J_F. If that is true, then $r(\cdot)$ is a Stackelberg strategy for the producer. The optimal consumption level is $\hat{q}$. Note that (5.4) and (5.5) can also be implemented by a two-part tariff with unit price p and fixed charge F, i.e., $r(q)=pq+F$, where

$$p=S(\bar{q}-\hat{q}),\quad F=k\hat{q}+\frac{1}{2}c\hat{q}^2-S\hat{q}(\bar{q}-\hat{q}).$$

In this case, the RSSP is linearly incentive-controllable.

VI. Incentives in a Divisionalized Firm

When A2) does not hold, i.e., the information structure is not nested, R1) becomes operative. In this case the method proposed in Section III-B cannot be used. However, a related team theoretic concept applies, as illustrated by the following example of organizational design.

Consider the problem of a firm with two more or less autonomous divisions A and B. Let the firm's payoff function be

$$J_L=J_A(u_A,\xi_A)+J_B(u_B,\xi_B) \tag{6.1}$$

where J_A (J_B) is the part of payoff gained by division $A(B)$; ξ_A (ξ_B) is a local parameter vector known only to A (B). The central headquarter needs to allocate some scarce resources u_A and u_B to divisions A and B. The choice of u_A and u_B is to be done so as to maximize J_L without direct

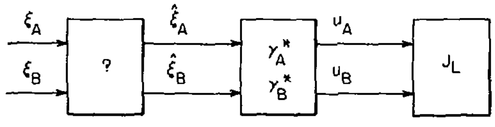

Fig. 3. Illustrating incentives in a divisionalized firm.

knowledge of ξ_A, ξ_B. Driven by selfish interests, division A may want to misrepresent ξ_A to the headquarter as $\hat{\xi}_A$, so as to increase J_A at the expense of J_B. (The same applies to B.) The problem here is to devise an incentive system so as to elicit honest responses. More specifically, the headquarter needs to decide, besides u_A and u_B, some bonuses u_{LA}, u_{LB} such that the payoff functions to divisions A and B are $\bar{J}_A=J_A+u_{LA}$, $\bar{J}_B=J_B+u_{LB}$, respectively. The basic idea works as follows.

1) Consider the problem $\text{Max}_{u_A,u_B}J_L$ with full knowledge of ξ_A and ξ_B. Let the solution be

$$u_A=\gamma_A{}^*(\xi_A,\xi_B),\quad u_B=\gamma_B{}^*(\xi_A,\xi_B). \tag{6.2}$$

Now suppose we implement $\gamma_A{}^*$, $\gamma_B{}^*$ and consider the related problem, as illustrated in Fig. 3.

We are asked the question: "If we observe ξ_A and ξ_B, then should we report the truth or something different to the box ($\gamma_A{}^*$, $\gamma_B{}^*$) in order to maximize J_L?" It is intuitively obvious that nothing can be gained by not telling the truth, and the box (?) is the identity function. The mechanism for eliciting honest response simply builds on this idea.[8]

2) Let the headquarter declare a bonus to division A such that the combined return to division A becomes

$$\bar{J}_A=J_A\left(\gamma_A{}^*(\hat{\xi}_A,\hat{\xi}_B),\xi_A\right)+J_B\left(\gamma_B{}^*(\hat{\xi}_A,\hat{\xi}_B),\hat{\xi}_B\right)+h_A(\hat{\xi}_B). \tag{6.3}$$

Then for the purpose of $\text{Max}_{\hat{\xi}_A}\bar{J}_A$, division A faces essentially the same problem as that of Fig. 3; hence, $\hat{\xi}_A=\xi_A$.[9] Note that (6.3) does not depend on ξ_B, but only on $\hat{\xi}_B$. Thus, true response is optimal for A even if he knows that B did not respond correctly.

3) The term $h_A(\hat{\xi}_B)$ is more or less arbitrary and can be chosen to satisfy

$$h_A(\hat{\xi}_B)+h_B(\hat{\xi}_A)+J_A(\gamma_A{}^*,\hat{\xi}_A)+J_B(\gamma_B{}^*,\hat{\xi}_B)<0 \tag{6.4}$$

so that the center always has the money to pay for the bonus. The ideas in 1) and 2) are more general than this specific application and can be considerably extended [6].

From the view of the Stackelberg problem with reversed information structure, the center declares to A the incentive policy

$$u_{LA}=J_B\left(\gamma_B{}^*(\hat{\xi}_A,\hat{\xi}_B),\hat{\xi}_B\right)+h_A(\hat{\xi}_B)$$

$$u_A=\gamma_A{}^*(\hat{\xi}_A,\hat{\xi}_B),\quad u_B=\gamma_B{}^*(\hat{\xi}_A,\hat{\xi}_B).$$

But A is required to act first by declaring $\hat{\xi}_A$ (and similarly for B) knowing the payoff to be $\bar{J}_A=J_A+u_{LA}$.

VII. Deterministic Multistage Dynamic Problems

The heart of the multistage dynamic Stackelberg problem is the single stage case just treated. The only difference is that the possibilities for different incentive or threat mechanism due to different information structures increase enormously. The problem treated in [2], [4], and [17] are only special cases. The surface has only been scratched. A separate and detailed discussion will be presented elsewhere [18]. However, the main idea can be sketched out here. First, we have decision variables $u_L(0), u_L(1),\cdots,u_L(t),\cdots,u_L(T-1)$ and $u_F(0), u_F(1),\cdots,u_F(t),\cdots,u_F(T-1)$, and the corresponding strategies. It is customary, although by no means necessary, to assume that

[8] Theorem 4.1.1 of [15].

[9] It is reasonable and accepted (see [15, page 189]) to assume that if a player is indifferent between truth telling and cheating he will tell the truth. Thus, we permit nonunique maxima in $\bar{J}_A$.

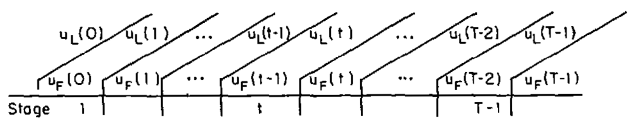

Fig. 4. Decomposition of a multistage problem.

A11):

$$\eta_L(t): x(0),\ldots,x(t), u_L(0),\ldots,u_L(t-1)$$
$$\eta_F(t): x(0),\ldots,x(t), u_F(0),\ldots,u_F(t-1),$$

i.e., each player has perfect memory of the state (x) and his own control history. The point here is that given $x(t)$, $u_L(t-1)$, $x(t-1)$, we can in general calculate $u_F(t-1)$ or vice versa [4]. As shown in Fig. 4, the leader at time t can choose his decision $u_L(t)$ based on $u_F(t-1)$ (or the entire past decisions of the follower), thus L essentially imposes a kind of reversed information structure on F. Note that whoever gets to declare his strategy first becomes the leader. This approach requires separate treatments at $t=T-1$ and 0 by solving $u_F(T-1)$ first which has a permanently optimal solution, and considering $u_F(-1)$ to be fixed at zero as is evident in the works of Basar and Tolwinski. Also, bear in mind that the distinction between closed-loop Stackelberg controls and Stackelberg feedback strategies [1] still exists. With this understanding, closed-loop Stackelberg strategy for linear quadratic deterministic problem can be solved using the basic idea discussed in Section III. It is clear that many strategies γ_L^s are possible due to the enormous flexibility here. For example, L's strategy may punish a nonrational behavior of F for one stage only (as in [4]), two stages, etc., or for the rest of the game (as in [2]). It is thus possible that different γ_L^s may enjoy various advantages.

VIII. Conclusion

In this paper we have identified two reasons why L may be interested in the decision of F. First, knowing F's strategy and his decision may enable L to infer the states of nature [R1)]. Second, F's decision may directly affect L's payoff [R2)]. We then discussed mechanisms by which L can induce F to behave cooperatively. In case R1) the mechanism is to transform F's payoff function so that it looks like L's own. In case R2) it is to make directly any choice of F's strategy other than the cooperative one unpalatable. In either case the crucial requirement is that we have the reversed information structure as defined in Section III. It serves as a unifying ingredient in diverse applications.

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A New Computational Method for Stackelberg and Min–Max Problems by Use of a Penalty Method

KIYOTAKA SHIMIZU AND EITARO AIYOSHI

***Abstract*—This paper is concerned with the Stackelberg problem and the min–max problem in competitive systems. The Stackelberg approach is applied to the optimization of two-level systems where the higher level determines the optimal value of its decision variables (parameters for the lower level) so as to minimize its objective, while the lower level minimizes its own objective with respect to the lower level decision variables under the given parameters. Meanwhile, the min–max problem is to determine a min–max solution such that a function maximized with respect to the maximizer's variables is minimized with respect to the minimizer's variables. This problem is also characterized by a parametric approach in a two-level scheme. New computational methods are proposed here; that is, a series of nonlinear programming problems approximating the original two-level problem by application of a penalty method to a constrained parametric problem in the lower level are solved iteratively. It is proved that a sequence of approximated solutions converges to the correct Stackelberg solution, or the min–max solution. Some numerical examples are presented to illustrate the algorithms.**

I. Introduction

This paper is concerned with the Stackelberg problem and the min–max problem in competitive systems.

The Stackelberg solution [14], [12], [13], [8] is the most rational one to answer a question: what will be the best strategy for Player 1 who knows Player 2's objective function and has to choose his strategy first, while Player 2 chooses his strategy after announcement of Player 1's strategy. A problem in the field of competitive economics is one such problem.

The min–max problem [3], [2], [4], [9], [6] is formulated so that a function, maximized with respect to the maximizer's variables, is minimized with respect to the minimizer's variables. The min–max solution is optimal for the minimizer against the worst possible case that might be taken by the opponent (the maximizer). Thus, the min–max concept plays an important role in game theory.

Many articles on the equilibrium solutions, such as the Nash solution and the saddle-point solution, have been published. However, the Stackel-

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