The MIMO Radar and Jammer Games

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***Abstract*—The interaction between a smart target and a smart MIMO radar is investigated from a game theory perspective. Since the target and the radar form an adversarial system, their interaction is modeled as a two-person zero-sum game. The mutual information criterion is used in formulating the utility functions. The unilateral, hierarchical, and symmetric games are studied, and the equilibria solutions are derived.**

***Index Terms*—Game theory, hierarchical game, jamming, MIMO radar, Nash equilibrium, Stackelberg equilibrium, waveform.**

I. Introduction

The success of the multiple-input multiple-output (MIMO) structure in communications has inspired investigation of MIMO radar. MIMO radars do not have a standard definition, and current literature divides them into statistical [1] and co-located [2], based on the antenna configuration. Generally, a statistical MIMO radar leverages the diversity of propagation path with sufficiently dissimilar transmitter-receiver geometry to improve detection, estimation, and information extraction [1], [3]–[8]; while a co-located one implies spatially coherent processing such as beamforming and direction-of-arrival estimation [2], [9]–[11]. The eventual acceptance of MIMO radar still remains unclear [12].

Waveform diversity is a key feature of a MIMO radar system [2]–[16]. It emphasizes illumination cooperation, and may provide an opportunity to upgrade radar performance. The specification of a waveform set largely depends on the system task. For propagation path separation, waveforms are required to be (near) orthogonal in order to avoid cross interference [3], [13]–[15]. In beampattern design, waveforms are correlated, so maximal transmission power can be focused in a certain direction [9]–[11]. In information extraction, the mutual information (MI) between the target response and collected echoes is maximized [4]–[8]. In target detection, optimized waveforms are designed to assure least likely missed detection for a given false alarm rate [8] or to maximize the signal-to-interference-plus-noise ratio [16]. And in target scatterer matrix estimation, waveforms are optimized for minimum mean square error [4]–[6].

Among those waveform design criteria, MI has acquired extensive attention. In the pioneering work [17], Woodward first suggested the application of information theory to radar receiver design. Later, Bell showed that maximizing the MI between target impulse response and measurement may enable the radar system a better capacity in characterizing the target in a contaminated environment [18]. Some interesting extensions including MI based waveform design in the presence of multiple targets [19], MI based MIMO radar space time code optimization [8] and waveform design [4]–[7] emerge thereafter. In this paper, we will concentrate on the application of the MI criterion to statistical MIMO radar.

Current literature on MIMO radar waveform design prefers to investigate the interaction between a *smart* radar and a *dumb* target, where the former has some knowledge of the latter such as radar cross section (RCS) distribution, while the latter is incapable of interfering with the former. Actually, with the development of electronic warfare, many noncooperative targets such as fighters are equipped with countermeasure systems to prevent a radar from operating as well as it might [20]. In this paper, the interaction involves a smart target, which carries jamming equipment that could intelligently confuse the radar. If the target always tries to prevent a radar from fulfilling its task, the interaction between them can be modeled as a two-person zero-sum (TPZS) game [21].

As in [4]–[8], the MI criterion is utilized to formulate the utility functions. The radar controls the waveform matrix to maximize the MI, while the latter has some access to its jamming matrix to minimize it. The contributions of this paper are as follows.

- We suggest the use of a MI based TPZS game to model the interaction between a target and MIMO radar, and categorize the game into one of three—unilateral, hierarchial, and symmetric—based on the information set available for each player.
- In the unilateral case, where one player can intercept the other's strategy while the latter *does not* notice that this is happening, the TPZS games are simplified as single person optimizations. For this case, the optimal (water-filling) strategies are derived.
- In the hierarchial case, where one player can intercept the other's strategy while the latter *does* notice that, the TPZS game is recast as a conservative minmax or maxmin two-stage optimization. The Stackelberg equilibria—optimization solutions—are derived.

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- We give the existence conditions and the expression of the Nash equilibrium for the symmetric game, where each player has no idea of the other's strategy. Some possible strategy combinations are discussed.

The application of game theory in radar is not new, and several interesting works can be found for both the monostatic and multistatic configurations. For example, a detection game between a target and a radar is analyzed under the Neyman-Pearson criterion in [22]; military air operations including missile guidance and air combat are explored in [23] and [24]; the tracking of an intelligent invader with a monostatic radar is investigated with the differential game theory in [25], and so forth. To the best of our knowledge, a game theory based analysis of the interaction between a target and MIMO radar, however, has not received attention yet.

Game theory is also used in communication to model multi-user interference [26]–[29]. For example, various two-user noncooperative games are summarized in [26], where each user considers the other as interference and greedily maximizes its individual capacity. In [27], the hostile interaction between a relay and jammer is investigated in the presence of receiver noise and a weak source–destination link. Reference [28] focuses on the interaction between a MIMO communication system and jammer, while some network layer considerations including competitive routing and contested spectrum are collected in [29]. Different from [26] and [27], where each user has only a single antenna, a player in this paper controls several. The transmitted information for each antenna is a modeled as a scalar in [28], while each antenna can send a waveform vector in our MIMO radar model. More important, as regards utility function formulation, a communication system concentrates on the MI between the received and transmitted signals [26]–[28], while a MIMO radar emphasizes the MI between the received signal and target response matrix (or channel matrix) [4]–[8]. More differences between them can be found in [6].

The remaining parts are organized as follows. Section II introduces the MIMO radar signal model. Section III specifies the MI game criterion. Section IV investigates unilateral games, while hierarchical games are in Section V. Section VI focuses on games with symmetric information. Section VII shows numerical results, and then conclusions are drawn.

Notation: Boldface uppercase and lowercase letters denote matrices and column vectors respectively; $\boldsymbol{I}_k$ indicates a $k \times k$ identity matrix; $(\cdot)^T$ and $(\cdot)^H$ denote transpose and Hermitian transpose, respectively; $\mathrm{diag}(\boldsymbol{a})$ denotes the diagonal matrix formed by vector $\boldsymbol{a}$, while $\det(\boldsymbol{A})$, $\mathrm{Tr}(\boldsymbol{A})$, and $\mathrm{rank}(\boldsymbol{A})$ obtain the determinant, trace, and rank of $\boldsymbol{A}$ respectively. $\mathbb{E}\{\cdot\}$ denotes the mathematical expectation. Finally, $\boldsymbol{a} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{A})$ means that $\boldsymbol{a}$ is a zero-mean complex Gaussian vector with covariance matrix $\boldsymbol{A}$.

II. MIMO Radar Signal Model

A. Jamming-Free Modeling Recap

A statistical MIMO radar employs distributed antennas to combat RCS fluctuation. Let the radar system be comprised of n_t transmitters and n_r receivers, all properly synchronized. Suppose that the transmitted waveform of the jth transmitter is $\boldsymbol{s}_j$, and then the collection of *jamming-free* echoes for the ith receiver is modeled as [1], [3]

$$\boldsymbol{y}_i = \boldsymbol{S}\boldsymbol{h}_i + \boldsymbol{w}_i \tag{1}$$

where

$\boldsymbol{h}_i = [h_{i,1}, h_{i,2}, \cdots, h_{i,n_t}]^T$	the path gain vector for receiver i;
$\boldsymbol{S} = [\boldsymbol{s}_1, \boldsymbol{s}_2, \cdots, \boldsymbol{s}_{n_t}]$	a $K \times n_t$ transmitted waveform matrix with $K \geq n_t$, and K denotes the waveform length;
$\boldsymbol{w}_i$	the $K \times 1$ receiver noise vector.

Define $\boldsymbol{H} = [\boldsymbol{h}_1, \boldsymbol{h}_2, \cdots, \boldsymbol{h}_{n_r}]$ as the $n_t \times n_r$ path gain matrix; therefore, the received signal matrix $\boldsymbol{Y} = [\boldsymbol{y}_1, \boldsymbol{y}_2, \cdots, \boldsymbol{y}_{n_r}]$ can be compactly written as

$$\boldsymbol{Y} = \boldsymbol{S}\boldsymbol{H} + \boldsymbol{W} \tag{2}$$

where $\boldsymbol{W} = [\boldsymbol{w}_1, \boldsymbol{w}_2, \cdots, \boldsymbol{w}_{n_r}]$ is the $K \times n_r$ noise matrix.

The signal model (2) is very general, and it can represent either a statistical or a co-located MIMO radar by properly formulating $\boldsymbol{H}$. In general, each element of $\boldsymbol{H}$, say $h_{i,j}$, stands for gain along a certain path: jth-receiver—target—ith-transmitter, and it integrates antenna steering vectors and bistatic radar reflection coefficient [1], [3]. The transmitters and receivers for a co-located MIMO radar are sufficiently close that all the propagation paths share the same random reflection coefficient ϱ. As a result, $\boldsymbol{H}$ becomes a rank-one matrix, and it can be decomposed as $\boldsymbol{H} = \varrho\boldsymbol{a}(\theta)\boldsymbol{b}(\theta)^T$, where $\boldsymbol{a}(\theta)$ and $\boldsymbol{b}(\theta)$ respectively denote the transmitting and receiving steering vectors [1]. A key problem for a co-located MIMO radar is how to improve beampattern design via a joint optimization of $\boldsymbol{a}(\theta)$ and $\boldsymbol{S}$; this is named as spatially coherent processing with waveform diversity [2], [9]–[11]. It is interesting to see that 1) a co-located MIMO radar concerns itself more with steering vectors rather than ϱ in coherent processing and 2) the dimension of its signal subspace is one as $\mathrm{rank}(\mathbb{E}\{\boldsymbol{S}\boldsymbol{H}\boldsymbol{H}^H\boldsymbol{S}^H\}) = 1$. As for a statistical MIMO radar, antennas are sufficiently separated, and $h_{i,j}$'s are assumed as independent random variables. Since beamforming is less interesting for a distributed configuration, there is no need to separately consider steering vectors and reflection coefficients. In addition, as the steering vectors only introduce phase shifts, the distribution of $\boldsymbol{H}$ is identical to that of target scattering matrix—a collection of reflection coefficients. So in some statistical MIMO radar publications, $\boldsymbol{H}$ is also equivalently termed as target scattering matrix [7]. With sufficient antenna span, $\boldsymbol{H}$ becomes a full rank random matrix [1], and the dimension of signal subspace turns to be n_t instead of 1. With knowledge of the statistics of $\boldsymbol{H}$, a statistical MIMO radar could adaptively allocate its waveform power within the noise space so as to enhance system performance [4]–[8]. Mathematically, such a noncoherent improvement with spatial diversity is attributed to the expansion of signal subspace.

A statistical MIMO radar system considers the radar-target interaction under a probability framework, and the distributions of $\boldsymbol{H}$ and $\boldsymbol{W}$ are the key issues. Three frequently utilized assumptions are as follows.

A1) The receivers are homogeneous, and $\boldsymbol{w}_i$'s are identically and independently distributed (i.i.d.) Gaussian vectors with distribution $\boldsymbol{w}_i \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{R}_w)$.

A2) The transmitters and receivers are sufficiently separated, and the propagation gains $h_{i,j}$'s for different bistatic geometries are independent. Furthermore, the target of interest is comprised of a large number of small i.i.d. random scatterers. With *the central limit theorem*, the $\boldsymbol{h}_i$'s could be considered as i.i.d. Gaussian vectors with distribution $\boldsymbol{h}_i \sim \mathcal{CN}(\boldsymbol{0}, \sigma_h^2 \boldsymbol{I}_{n_t})$ [3].

A3) $\boldsymbol{H}$ and $\boldsymbol{W}$ are mutually independent, and neither of them relies on waveform matrix $\boldsymbol{S}$.

Note that (2) plus these three assumptions compose a classical model for the statistical MIMO radar, many interesting results on target detection, estimation, information extraction, and waveform optimization rely on it [3]–[8].

B. Jamming Modeling

In general, the target and radar system are noncooperative. Survival requires countermeasures, and hence there has been considerable research on jamming [20], [22]. Electronic jamming interferes with the operation of a radar by saturating its receiver with electromagnetic noise or false information, and it has many realizations such as spot, sweep, barrage and deceptive [20]. In the following, we consider a typical approach—*barrage jamming*, where the target interferes with the MIMO radar system via waveform independent noise. Mathematically, the MIMO radar signal model could be modified as

$$\boldsymbol{Y} = \boldsymbol{S}\boldsymbol{H} + \boldsymbol{J} + \boldsymbol{W} \tag{3}$$

where $\boldsymbol{J}$ denotes the $K \times n_r$ jamming matrix. Here, one more assumption is appended:

A4) $\boldsymbol{J}$ is independent of $\boldsymbol{H}$ and $\boldsymbol{W}$, and its columns are i.i.d. random vectors with distribution $\mathcal{CN}(\boldsymbol{0}, \boldsymbol{R}_b)$. We use the subscript 'b' for barrage.

The relationship between the two game players, radar and target, is formulated. The former controls the waveform matrix $\boldsymbol{S}$, and the latter dominates $\boldsymbol{J}$. This is a TPZS game [21], where one player's gain is the other's loss.

III. Mutual Information

A. MI Formulation

As in [4]–[8], the MI criterion is adopted in this paper. The MIMO radar wants to extract MI between the received signal $\boldsymbol{Y}$ and the path gain matrix $\boldsymbol{H}$

$$I(\boldsymbol{Y};\boldsymbol{H}|\boldsymbol{S}) = h(\boldsymbol{Y}|\boldsymbol{S}) - h(\boldsymbol{J}+\boldsymbol{W}) \tag{4}$$

in a contaminated environment, where $h(\boldsymbol{Y}|\boldsymbol{S})$ represents the conditional differential entropy, and it is written as [30]

$$\begin{aligned} h(\boldsymbol{Y}|\boldsymbol{S}) &= -\int f(\boldsymbol{Y},\boldsymbol{S}) \log f(\boldsymbol{Y}|\boldsymbol{S}) d\boldsymbol{Y} d\boldsymbol{S} \\ &= -\int f(\boldsymbol{Y}|\boldsymbol{S}) \log f(\boldsymbol{Y}|\boldsymbol{S}) d\boldsymbol{Y} \\ &= -\mathbb{E}\{\log f(\boldsymbol{Y}|\boldsymbol{S})\} \end{aligned} \tag{5}$$

because $\boldsymbol{S}$ is nonrandom. As the conditional probability density function (pdf) of $\boldsymbol{Y}$ for a given $\boldsymbol{S}$ is

$$\begin{aligned} f(\boldsymbol{Y}|\boldsymbol{S}) &= \frac{1}{\pi^{n_r K} \det^{n_r}\left(\sigma_h^2 \boldsymbol{S}\boldsymbol{S}^H + \boldsymbol{R}_b + \boldsymbol{R}_w\right)} \\ &\quad \times \exp\left\{-\mathrm{Tr}\left[\left(\sigma_h^2 \boldsymbol{S}\boldsymbol{S}^H + \boldsymbol{R}_b + \boldsymbol{R}_w\right)^{-1} \boldsymbol{Y}\boldsymbol{Y}^H\right]\right\}, \end{aligned} \tag{6}$$

the conditional differential entropy $h(\boldsymbol{Y}|\boldsymbol{S})$ is recast as

$$\begin{aligned} h(\boldsymbol{Y}|\boldsymbol{S}) =& \mathbb{E}\{n_r K \log \pi\} \\ &+ \mathbb{E}\left\{n_r \log\det\left(\sigma_h^2 \boldsymbol{S}\boldsymbol{S}^H + \boldsymbol{R}_b + \boldsymbol{R}_w\right)\right\} \\ &+ \mathbb{E}\left\{\mathrm{Tr}\left[\left(\sigma_h^2 \boldsymbol{S}\boldsymbol{S}^H + \boldsymbol{R}_b + \boldsymbol{R}_w\right)^{-1} \boldsymbol{Y}\boldsymbol{Y}^H\right]\right\}. \end{aligned} \tag{7}$$

Since the first two items are independent of $\boldsymbol{Y}$, and since

$$\begin{aligned} &\mathbb{E}\left\{\mathrm{Tr}\left[\left(\sigma_h^2 \boldsymbol{S}\boldsymbol{S}^H + \boldsymbol{R}_b + \boldsymbol{R}_w\right)^{-1} \boldsymbol{Y}\boldsymbol{Y}^H\right]\right\} \\ &= \mathrm{Tr}\left[\left(\sigma_h^2 \boldsymbol{S}\boldsymbol{S}^H + \boldsymbol{R}_b + \boldsymbol{R}_w\right)^{-1} \mathbb{E}\{\boldsymbol{Y}\boldsymbol{Y}^H\}\right] \\ &= n_r \mathrm{Tr}[\boldsymbol{I}_K] \end{aligned} \tag{8}$$

we have

$$h(\boldsymbol{Y}|\boldsymbol{S}) = c + n_r \log\left[\det\left(\sigma_h^2 \boldsymbol{S}\boldsymbol{S}^H + \boldsymbol{R}_b + \boldsymbol{R}_w\right)\right] \tag{9}$$

where $c \triangleq n_r K \log \pi + n_r K$ is a constant. Moreover, since

$$\begin{aligned} f(\boldsymbol{J}+\boldsymbol{W}) &= \frac{1}{\pi^{n_r K} \det^{n_r}(\boldsymbol{R}_b + \boldsymbol{R}_w)} \\ &\quad \times \exp\left\{-\mathrm{Tr}\left[(\boldsymbol{R}_b + \boldsymbol{R}_w)^{-1} (\boldsymbol{J}+\boldsymbol{W})(\boldsymbol{J}+\boldsymbol{W})^H\right]\right\}, \end{aligned} \tag{10}$$

with a similar derivation, we have

$$h(\boldsymbol{J}+\boldsymbol{W}) = c + n_r \log\left[\det(\boldsymbol{R}_b + \boldsymbol{R}_w)\right]. \tag{11}$$

Substituting (9) and (11) into (4), we obtain

$$I_b \triangleq I(\boldsymbol{Y};\boldsymbol{H}|\boldsymbol{S}) = n_r \log \frac{\det\left(\sigma_h^2 \boldsymbol{S}\boldsymbol{S}^H + \boldsymbol{R}_b + \boldsymbol{R}_w\right)}{\det(\boldsymbol{R}_b + \boldsymbol{R}_w)}. \tag{12}$$

Apparently, I_b is a function of the radar-target strategy pair $(\boldsymbol{S}, \boldsymbol{R}_b)$. The optimal strategy of one player depends on its *inference* of the other's.

B. Eigenspace Selection

The strategy domain of (12) is composed of two Hermitian matrices: $\boldsymbol{R}_b$ and $\boldsymbol{S}\boldsymbol{S}^H$. A direct interaction analysis in matrix domain is rather complex, particularly when one player has no knowledge of the other. As a Hermitian matrix is determined by its eigenvectors and eigenvalues, the TPZS game actually implies two parts: eigenspace selection (where to play) and eigenvalue optimization (how to allocate power). This subsection concentrates on the first. Here we treat the question whether each player has an eigenspace preference.

Proposition 1: Let the eigendecomposition of $\boldsymbol{R}_w$, $\boldsymbol{R}_b$, and $\boldsymbol{S}\boldsymbol{S}^H$ be $\boldsymbol{R}_w = \boldsymbol{U}_w \boldsymbol{\Lambda}_w \boldsymbol{U}_w^H$, $\boldsymbol{R}_b = \boldsymbol{U}_b \boldsymbol{\Lambda}_b \boldsymbol{U}_b^H$, and $\boldsymbol{S}\boldsymbol{S}^H = \boldsymbol{U}_s \boldsymbol{\Gamma}_s \boldsymbol{U}_s^H$ respectively, and then

$$\boldsymbol{U}_s = \boldsymbol{U}_w \boldsymbol{P}_1 \quad \text{and} \quad \boldsymbol{U}_b = \boldsymbol{U}_w \boldsymbol{P}_2 \tag{13}$$

compose an equilibrium in eigenspace selection, where $\boldsymbol{P}_1$ and $\boldsymbol{P}_2$ are two arbitrary permutation matrices. Furthermore, this equilibrium is, and the only one is, invariant to the specifications of $\boldsymbol{\Lambda}_b$ and $\boldsymbol{\Gamma}_s$.

Proof: Proof is in Appendix B. ■

In Proposition 1, the eigenspace equilibrium means that if the target chooses $\boldsymbol{U}_w \boldsymbol{P}_2$ to span its jamming covariance matrix, the best response eigenspace for radar is $\boldsymbol{U}_w \boldsymbol{P}_1$, and vice versa. However, no one could guarantee that two players converge to a given equilibrium unless it is unique or it has other unique attractive properties among all the equilibria [21]. Even though $(\boldsymbol{U}_s = \boldsymbol{U}_w \boldsymbol{P}_1, \boldsymbol{U}_b = \boldsymbol{U}_w \boldsymbol{P}_2)$ may be not unique, it is the only one that is invariant to power allocation specifications. In other words, no matter how the two players allocate their power, the eigenspace equilibrium will still hold. As a result, if one player could not precisely infer the other's power allocation, staying at the eigenspace defined in Proposition 1 would be a secure choice.

Based on the analysis in the previous paragraph, another assumption is inserted:

A5) The radar and target choose the eigenspace equilibrium defined in Proposition 1, and the eigenvectors spanning the jamming and waveform space are the same as those for the noise space.

Since the dimension of signal subspace is n_t, the eigenvalue matrix $\boldsymbol{\Gamma}_s$ can be written as

$$\boldsymbol{\Gamma}_s = \begin{bmatrix} \boldsymbol{\Lambda}_s & \\ & \boldsymbol{0}_{(K-n_t)\times(K-n_t)} \end{bmatrix} \tag{14}$$

where $\boldsymbol{\Lambda}_s = \text{diag}([\sigma_1^s, \sigma_2^s, \cdots, \sigma_{n_t}^s])$. Let the diagonal elements of $\boldsymbol{\Lambda}_w = \text{diag}([\sigma_1^w, \sigma_2^w, \cdots, \sigma_K^w])$ be in decreasing order

$$\sigma_1^w \geq \sigma_2^w \geq \cdots \geq \sigma_K^w \tag{15}$$

while those of $\boldsymbol{\Lambda}_b = \text{diag}([\sigma_1^b, \sigma_2^b, \cdots, \sigma_K^b])$ do not have any ordering requirement. Without loss of generality, define the waveform matrix as $\boldsymbol{S} \triangleq \boldsymbol{U}_w \boldsymbol{P}_1 [\sqrt{\boldsymbol{\Lambda}_s}, \boldsymbol{0}_{n_t \times (K-n_t)}]^T$, and then the MI at the equilibrium is specified as

$$\bar{I}_b = n_r \log \left[\det \left(\sigma_h^2 \boldsymbol{\Gamma}_s \boldsymbol{P}_1 (\boldsymbol{\Lambda}_b + \boldsymbol{\Lambda}_w)^{-1} \boldsymbol{P}_1^T + \boldsymbol{I}_K \right) \right] \tag{16}$$

where the extended diagonal matrix $\boldsymbol{\Gamma}_s$ will reduce the dimension of the game space from K to n_t, and $\boldsymbol{P}_1$ decides which subspace would be selected. Now, the waveform and jamming design problem reveals itself as a power allocation TPZS game, where the radar system and target explore their strategies respectively to maximize and minimize $\bar{I}_b$.

IV. Unilateral Games

This section considers the extreme cases, where one player has complete knowledge of the other's strategy. Therefore, that player can always choose the best response, and the game reduces to a unilateral optimization.

A. Radar Unilateral Games

If the MIMO radar knows the target's strategy, the game degenerates to a classical power allocation problem [4], [7], where the radar assigns its power into the noise (jamming) space to maximize the MI. Mathematically, it is formulated as

$$\max_{\boldsymbol{\Lambda}_s, \boldsymbol{P}_1} \bar{I}_b, \text{ s.t. } \text{Tr}(\boldsymbol{S}\boldsymbol{S}^H) = \text{Tr}(\boldsymbol{\Lambda}_s) \leq P_s \tag{17}$$

where P_s denotes the available waveform power. Without loss of generality, we assume $\sigma_1^b + \sigma_1^w \geq \sigma_2^b + \sigma_2^w \geq \cdots \geq \sigma_K^b + \sigma_K^w$ and $\sigma_1^s \geq \sigma_2^s \geq \cdots \geq \sigma_{n_t}^s$ in this subsection. Based on Lemma 2 in Appendix A, (17) is maximized if $\boldsymbol{P}_1$ is chosen as $\boldsymbol{P}_1 = \boldsymbol{P}$, where $\boldsymbol{P}$ is a skew identity matrix

$$\boldsymbol{P} = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix}, \tag{18}$$

and yields

$$\begin{aligned} \max_{\sigma_i^s} \quad & \sum_{i=1}^{n_t} \log \left(\frac{\sigma_i^s \sigma_h^2}{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w} + 1 \right) \\ \text{s.t.} \quad & \sum_{i=1}^{n_t} \sigma_i^s \leq P_s. \end{aligned} \tag{19}$$

The above optimization involves a concave and monotonic increasing objective function and linear constraints. Its optimal solution can be obtained via Lagrange multipliers, and yields a water-filling strategy [4], [7]

$$\sigma_i^s = \left(\lambda_1 - \frac{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w}{\sigma_h^2} \right)^+ \tag{20}$$

where $(x)^+ \triangleq \max\{0, x\}$, and $\lambda_1 > 0$ is chosen implicitly via $\sum_{i=1}^{n_t} \sigma_i^s = P_s$. Obviously, the optimal strategy distributes more power for an activated subspace—where $\sigma_i^s > 0$—with a lower $(\sigma_{K+1-i}^b + \sigma_{K+1-i}^w)$ value.

B. Target Unilateral Games

On the other hand, suppose that the target knows the power allocation strategy of the MIMO radar. The game degenerates to a jamming unilateral optimization, as the radar is not aware of this. In such a circumstance, the target will allocate its jamming power to minimize $\bar{I}_b$. Mathematically, this is expressed as

$$\min_{\mathbf{\Lambda}_b} \bar{I}_b, \text{ s.t. } \mathrm{Tr}(\mathbf{\Lambda}_b) \le P_b \tag{21}$$

where P_b bounds the jamming power. For a given radar power allocation strategy, $\bar{\boldsymbol{P}}_1$ plus $\bar{\mathbf{\Lambda}}_s = \mathrm{diag}([\bar{\sigma}_1^s, \bar{\sigma}_2^s, \cdots, \bar{\sigma}_{n_t}^s])$, the optimization (21) is specified as

$$\min_{\bar{\sigma}_i^b} \sum_{i=1}^{n_t} \log\left(\frac{\bar{\sigma}_i^s \sigma_h^2}{\bar{\sigma}_i^b + \bar{\sigma}_i^w} + 1\right), \quad \text{s.t.} \sum_{i=1}^{n_t} \bar{\sigma}_i^b \le P_b \tag{22}$$

where $\bar{\sigma}_i^b$ and $\bar{\sigma}_i^w$ correspond to the ith selected jamming-noise subspace, and they are not necessarily identical to σ_i^b and σ_i^w.

The payoff function of (22) is a summation of n_t separable subfunctions

$$f_1\left(\bar{\sigma}_i^b | \bar{\sigma}_i^w\right) = \log\left(\frac{a_i}{\bar{\sigma}_i^b + \bar{\sigma}_i^w} + 1\right) \tag{23}$$

where $a_i \triangleq \bar{\sigma}_i^s \sigma_h^2 > 0$ for notational simplicity. Since

$$\begin{aligned} \frac{\partial f_1(\cdot)}{\partial \bar{\sigma}_i^b} &= \frac{-a_i}{a_i\left(\bar{\sigma}_i^w + \bar{\sigma}_i^b\right) + \left(\bar{\sigma}_i^w + \bar{\sigma}_i^b\right)^2} < 0 \\ \frac{\partial^2 f_1(\cdot)}{\partial^2 \bar{\sigma}_i^b} &= \frac{a_i^2 + 2a_i\left(\bar{\sigma}_i^w + \bar{\sigma}_i^b\right)}{\left[a_i\left(\bar{\sigma}_i^w + \bar{\sigma}_i^b\right) + \left(\bar{\sigma}_i^w + \bar{\sigma}_i^b\right)^2\right]^2} > 0, \end{aligned} \tag{24}$$

$f_1(\bar{\sigma}_i^b | \bar{\sigma}_i^w)$ is monotonically decreasing and strictly convex. Therefore, (22) has a unique optimal solution and can be found with Lagrange multipliers [31]

$$L = \sum_{i=1}^{n_t} \log\left(\frac{\bar{\sigma}_i^s \sigma_h^2}{\bar{\sigma}_i^b + \bar{\sigma}_i^w} + 1\right) + \lambda_2\left(\sum_{i=1}^{n_t} \bar{\sigma}_i^b - P_b\right). \tag{25}$$

Differentiating L with respect to $\bar{\sigma}_i^b$ and setting the result to zero, we get two solutions

$$\bar{\sigma}_i^b = \pm\sqrt{\frac{\bar{\sigma}_i^s \sigma_h^2}{\lambda_2} + \frac{\left(\bar{\sigma}_i^s \sigma_h^2\right)^2}{4}} - \bar{\sigma}_i^w - \frac{\bar{\sigma}_i^s \sigma_h^2}{2}. \tag{26}$$

Deleting the negatives, the optimal one remains

$$\bar{\sigma}_i^b = \left(\sqrt{\frac{\bar{\sigma}_i^s \sigma_h^2}{\lambda_2} + \frac{\left(\bar{\sigma}_i^s \sigma_h^2\right)^2}{4}} - \bar{\sigma}_i^w - \frac{\bar{\sigma}_i^s \sigma_h^2}{2}\right)^+. \tag{27}$$

where $\lambda_2 > 0$ satisfies $\sum_{i=1}^{n_t} \bar{\sigma}_i^b = P_b$. Since the subspace selection privilege belongs to the radar, the target can only optimize its power corresponding to the selected collection.

V. Hierarchical Games

The previous section investigated games with asymmetrical information. Here, we are interested in the hierarchical case [21], where the inferior player, called the *leader* in a game, knows that its strategy will be intercepted by its opponent. With conservativeness and rationality assumptions, the leader may adopt the strategy which can alleviate the worst case, and the game yields a Stackelberg equilibrium (SE) [21].

A. Target As The Leader

Let the radar system possess sufficient interception capacity that it can immediately sense interference. If the target *does* know this and behaves conservatively, the game may converge to a SE, which is defined as the solution of a two-stage optimization [21]

$$\begin{aligned} &\min_{\mathbf{\Lambda}_b} \max_{\mathbf{\Lambda}_s, \boldsymbol{P}_1} \quad \log\left[\det\left(\sigma_h^2 \mathbf{\Gamma}_s \boldsymbol{P}_1 (\mathbf{\Lambda}_b + \mathbf{\Lambda}_w)^{-1} \boldsymbol{P}_1^T + \boldsymbol{I}_K\right)\right] \\ &\quad \text{s.t.} \quad \mathrm{Tr}(\mathbf{\Lambda}_b) \le P_b, \mathrm{Tr}(\mathbf{\Lambda}_s) \le P_s. \end{aligned} \tag{28}$$

The interception capability enables the radar to optimally respond to its opponent, so the results in Section IV-A are still applicable for the first stage. Based on (20), (28) is reduced to

$$\begin{aligned} \min_{\sigma_i^b} \quad & \sum_{i=1}^{n_t} \log\left(\frac{\sigma_i^s \sigma_h^2}{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w} + 1\right) \\ \text{s.t.} \quad & \sigma_i^s = \left(\lambda_1 - \frac{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w}{\sigma_h^2}\right)^+, \\ & \sigma_1^b + \sigma_1^w \ge \sigma_2^b + \sigma_2^w \ge \cdots \ge \sigma_{K-n_t+1}^b + \sigma_{K-n_t+1}^w, \\ & \sum_{i=1}^{n_t} \sigma_i^s = P_s, \sum_{i=1}^{K} \sigma_i^b \le P_b. \end{aligned} \tag{29}$$

We emphasize that the second constraint is about the ordering of noise-jamming power, and it is responsible for the optimal subspace selection. Due to the fact that

$$\begin{aligned} & \log\left(\frac{\sigma_i^s \sigma_h^2}{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w} + 1\right) \\ \Longleftrightarrow\ & \log\left[\frac{\left(\lambda_1 - \left(\sigma_{K+1-i}^b + \sigma_{K+1-i}^w\right)/\sigma_h^2\right)^+ \sigma_h^2}{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w} + 1\right] \\ \Longleftrightarrow\ & \log\left[\left(\frac{\sigma_h^2 \lambda_1}{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w} - 1\right)^+ + 1\right] \\ \Longleftrightarrow\ & \log\left[\max\left(\frac{\sigma_h^2 \lambda_1}{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w}, 1\right)\right] \\ \Longleftrightarrow\ & \max\left[\log\left(\frac{\sigma_h^2 \lambda_1}{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w}\right), 0\right] \end{aligned} \tag{30}$$

where "$\Longleftrightarrow$" denotes the sign of mathematical equivalence, (29) could be simplified to

$$\begin{aligned} \min_{\sigma_i^b} \quad & \sum_{i=1}^{n_t} \left[\log \lambda_1 - \log \frac{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w}{\sigma_h^2}\right]^+ \\ \text{s.t.} \quad & \sum_{i=1}^{n_t} \left(\lambda_1 - \frac{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w}{\sigma_h^2}\right)^+ = P_s, \\ & \sum_{i=1}^{n_t} \sigma_i^b \le P_b - P_c(\lambda_1) \end{aligned} \tag{31}$$

where $P_c(\lambda_1)$ denotes the minimum power allocated to σ_j^b, where $1 \le j \le K - n_t$, to guarantee the ordering constraint. It is interesting to see that the corresponding items $(\cdot)^+$ in the objective function and constraint are always active simultaneously, due to the fact that the $\log(\cdot)$ operation preserves

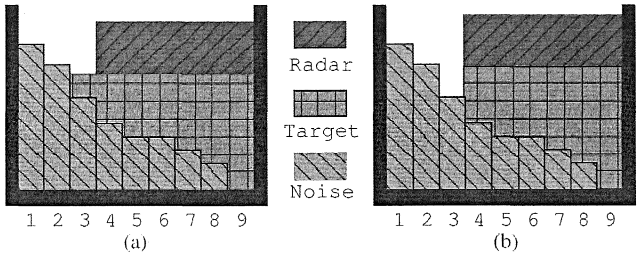

Fig. 1. An intuitive explanation of the SEs for the two hierarchical games: (a) target as the leader, and (b) radar as the leader. The dimension of noise subspace is $K = 9$, while that of the signal subspace is $n_t = 6$. As for case (a), since the target moves first and does not know which amongst the n_t subspaces will the radar choose, it has to allocate its power to the entire noise space (water-filling). As for case (b), target moves late and can observe which subspaces radar selected, so it only (water-filling) allocates its power to the radar-selected-ones: 4–9.

mathematical monotonicity. Now, (31) depends only on the σ_i^b's; one may solve it to find the equilibrium.

Proposition 2: The power allocation SE for the hierarchical game with the target as the leader is

$$\begin{aligned}\sigma_i^b &= (\lambda_3 - \sigma_i^w)^+, \text{for } 1 \le i \le K\\ \sigma_j^s &= \left(\min\left\{\lambda_1 - \lambda_3/\sigma_h^2, \lambda_1 - \sigma_{K+1-j}^w/\sigma_h^2\right\}\right)^+,\\ &\qquad \text{for } 1 \le j \le n_t \quad (32)\end{aligned}$$

where λ_3 and λ_1 are determined by $\sum_{i=1}^K \sigma_i^b = P_b$ and $\sum_{i=j}^{n_t} \sigma_j^s = P_s$.

Proof: Proof is in Appendix C. ■

Intuitively, the SE can be interpreted as a two-step water-filling as shown in Fig. 1(a): first, the target conservatively fills its jamming power to the noise space, and then the radar injects its power to jamming-plus-noise space. The uniqueness of the SE depends on the number of maxmin solutions. If $K = n_t$, Proposition 2 leads to exactly one solution; therefore, the SE is unique. Define the jamming power threshold P_n as

$$P_n \triangleq \sum_{i=1}^{n_t} \left(\sigma_{K-n_t}^w - \sigma_{K+1-i}^w\right) \quad (33)$$

for $K > n_t$. It is easy to check that the solution of (32) is unique for $P_b < P_n$, and thus so is the SE. However, if $P_b \ge P_n$, the SE will have multiple possibilities. For instance, let $K = 3, n_t = 2$, $\sigma_1^w = 2, \sigma_2^w = \sigma_3^w = 1, P_b = 2$, and $P_s = 2$, the power allocation SEs are $(\boldsymbol{\Lambda}_b = \text{diag}([0,1,1]), \boldsymbol{\Gamma}_s = \boldsymbol{P}_1\text{diag}([1,1,0])\boldsymbol{P}_1^T)$, where $\boldsymbol{P}_1$ denotes an arbitrary 3×3 permutation matrix. The multiple solutions property of (32) is due to the ordering expression of $(\sigma_i^b + \sigma_i^w)$'s, the second constraint in (29), is not unique. Interestingly, all the SEs share the same MI value. No matter whether the SE is unique, this game is guaranteed to go to one of them.

B. Radar As The Leader

Let the target be able to sense the radar's power allocation, and let the MIMO radar know that it does. Then a conservative radar system may select its strategy based on

$$\begin{aligned}&\max_{\boldsymbol{\Lambda}_s, \boldsymbol{P}_1} \min_{\boldsymbol{\Lambda}_b} \quad \log\left[\det\left(\sigma_h^2 \boldsymbol{\Gamma}_s \boldsymbol{P}_1 (\boldsymbol{\Lambda}_b + \boldsymbol{\Lambda}_w)^{-1} \boldsymbol{P}_1^T + \boldsymbol{I}_K\right)\right]\\ &\text{s.t.} \quad \text{Tr}(\boldsymbol{\Lambda}_b) \le P_b,\ \text{Tr}(\boldsymbol{\Lambda}_s) \le P_s \qquad (34)\end{aligned}$$

in order to optimize the worst case. As for (34), the first stage includes an unknown subspace selection parameter $\boldsymbol{P}_1$; direct optimization is hard. But from rationality considerations we know the radar system will not 'pour' its power to the $(K - n_t)$ subspaces with higher noise levels, because that will make the final result even worse. Let the radar choose the n_t noise subspaces corresponding to eigenvalues σ_{K+1-i}^w's, where $1 \le i \le n_t$. Without losing generality, one possible choice is $\boldsymbol{P}_1 = \boldsymbol{P}$. As a result, (34) is recast as

$$\begin{aligned}&\max_{\sigma_i^s} \min_{\sigma_i^b} \sum_{i=1}^{n_t} \log\left(\frac{\sigma_i^s \sigma_h^2}{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w} + 1\right),\\ &\text{s.t.} \quad \sum_{i=1}^{n_t} \sigma_{K+1-i}^b \le P_b, \sum_{i=1}^{n_t} \sigma_i^s \le P_s. \qquad (35)\end{aligned}$$

In additional to the optimization ordering, there is another significant difference between (29) and (35): the jamming power constraints. In the case of (29), the target moves first. It will conservatively fill its power to the entire noise space; therefore, the power constraint is $\sum_{i=1}^K \sigma_K^b \le P_b$. In the case of (35) the radar moves first, so the target can 'see' which subspaces are selected, and then it will pour the jamming energy only to them. Hence, the power constraints are modified to $\sum_{i=1}^{n_t} \sigma_{K+1-i}^b \le P_s$ in (35), and this can be regarded as a game in a reduced space.

A trivial optimization order swap makes the equilibrium analysis even more complicated, since the water-filling solution of the first stage involves a nonlinear expression of σ_i^s's as shown in (27). Fortunately, it can be indirectly solved with the help of *Sion's minimax theorem* [32].

Proposition 3: The optimization problem in (35) can be equivalently reformulated as

$$\begin{aligned}&\min_{\sigma_i^b} \max_{\sigma_i^s} \sum_{i=1}^{n_t} \log\left(\frac{\sigma_i^s \sigma_h^2}{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w} + 1\right),\\ &\text{s.t.} \quad \sum_{i=1}^{n_t} \sigma_{K+1-i}^b \le P_b, \sum_{i=1}^{n_t} \sigma_i^s \le P_s. \qquad (36)\end{aligned}$$

Proof: It is clear that the domains of $\boldsymbol{\Lambda}_s$ and $\boldsymbol{\Lambda}_b$ are both linear, compact, and convex. Moreover, the objective function is continuous, differentiable, and *quasilinear* (both quasi-concave and quasi-convex) [31]. Based on Lemma 4 in Appendix A, the proposition can be proven. ■

Proposition 4: The power allocation SE for the hierarchical game with the radar as the leader is

$$\begin{aligned}\sigma_i^b &= 0, \text{ for } 1 \le i \le K - n_t\\ \sigma_i^b &= (\lambda_4 - \sigma_i^w)^+, \text{ for } K - n_t + 1 \le i \le K\\ \sigma_j^s &= \left(\min\left\{\lambda_5 - \lambda_4/\sigma_h^2, \lambda_5 - \sigma_{K+1-j}^w/\sigma_h^2\right\}\right)^+,\\ &\quad \text{for } 1 \le j \le n_t \qquad (37)\end{aligned}$$

where λ_4 and λ_5 are determined by $\sum_{i=1}^{n_t} \sigma_{K+1-i}^b = P_b$ and $\sum_{i=1}^{n_t} \sigma_i^s = P_s$.

Proof: Following the proof of Proposition 2, Proposition 4 can be proven. ■

The SE is still a two-step water-filling in a reduced space; an illustration is depicted in Fig. 1(b). If $K = n_t$, the SE is unique because (37) has exactly one solution. If $K > n_t$ and $\sigma^w_{K-n_t} = \sigma^w_{K-n_t+1}$, the radar can choose the subspace corresponding to either $\sigma^w_{K-n_t}$ or $\sigma^w_{K-n_t+1}$ if its power satisfies $P_s > P_n - P_b$; therefore, the SE is no longer unique. For example, let $K = 3$, $n_t = 2$, $\sigma^w_1 = \sigma^w_2 = 3$, $\sigma^w_3 = 1$, $P_b = 1$, and $P_s = 3$. Both $(\mathbf{\Lambda}_b = \text{diag}([0,0,1]), \mathbf{\Gamma}_s = \text{diag}([1,0,2]))$ and $(\mathbf{\Lambda}_b = \text{diag}([0,0,1]), \mathbf{\Gamma}_s = \text{diag}([0,1,2]))$ are SEs. These SEs will not change the final MI result, and the game will go to one of them.

C. Discussion

The equivalence of *Propositions 2* and *4* is straightforward if $K = n_t$. This subsection discusses their relationship for $K > n_t$. Comparing (32) and (37), we see that if and only if (iff) P_b is large enough to activate the noise subspace corresponding to $\sigma^b_{K-n_t}$, say $P_b > P_n$, the two propositions will result in different strategy pairs, and it is interesting that the power allocation strategies of the radar are identical in both cases. This can be explained from two perspectives: 1) if $P_b \le P_n$, the two propositions are the same, so σ^s_j's are as well; 2) if $\sigma^b_{K-n_t} > 0$, we must have

$$\sigma^b_i + \sigma^w_i = \lambda_3 \ (\text{or } \lambda_4), \ \text{for } K - n_t + 1 \le i \le K \tag{38}$$

for both of them. Even though $\lambda_3 \neq \lambda_4$, they will both induce a uniform power allocation in the second step, and hence the strategies remain identical. An immediate corollary of this phenomenon is that the power allocation of the MIMO radar becomes uniform with the increase of P_b for both games, because the jamming-plus-noise subspaces, $(\sigma^b_i+\sigma^w_i)$'s, tend toward flat as shown in (38).

Another interesting question for hierarchical games is which role, leader or follower, is better. If the minmax and maxmin optimizations converge to the same solution, the two roles are equivalent as $\bar{I}^{\min\max}_b = \bar{I}^{\max\min}_b$. However, if they induce different power allocation strategies, we have $\bar{I}^{\min\max}_b > \bar{I}^{\max\min}_b$, which means that it is better to be a follower for both the target and the radar. In brief, the follower is not worse than the leader; however, that whether the former is better than the latter depends on the jamming power P_b and the dimension of signal subspace n_t.

In the following, the phrase *minmax (maxmin) game* is employed for simplicity to indicate a hierarchical game with radar (target) as leader.

VI. Games With Symmetric Information

A. Nash Equilibrium

In the unilateral and hierarchical games, the available information for the two "players" is asymmetric. This section studies the cases with symmetric information, where neither has knowledge of the other's strategy. In such circumstances, the Nash equilibrium (NE) is a good tool to analyze the outcome of the strategic interaction [21]. If a game is competitive and has a unique pure-strategy NE, all the players prefer to stay at NE under the assumptions of conservativeness and rationality. As for a TPZS game with utility function $f(\boldsymbol{a}, \boldsymbol{b})$, where $\boldsymbol{a}$ is a minimizer and $\boldsymbol{b}$ is a maximizer, the pure-strategy NE $(\boldsymbol{a}^*, \boldsymbol{b}^*)$ is defined as [21]

$$f(\boldsymbol{a}, \boldsymbol{b}^*) \ge f(\boldsymbol{a}^*, \boldsymbol{b}^*) \ge f(\boldsymbol{a}^*, \boldsymbol{b}), \ \text{for } \forall \boldsymbol{a} \neq \boldsymbol{a}^* \text{ and } \boldsymbol{b} \neq \boldsymbol{b}^*. \tag{39}$$

Informally speaking, the NE of a TPZS game on a continuous space is the saddlepoint of its utility function, and no player can do better by unilateral deviation. We need the following proposition.

Proposition 5: Let $f(\boldsymbol{a}, \boldsymbol{b})$ be a real valued function for a TPZS game, where $\boldsymbol{a} \in \mathcal{A}$ is a minimizer and $\boldsymbol{b} \in \mathcal{B}$ is a maximizer. Suppose $\bar{\mathcal{A}} \times \bar{\mathcal{B}} \neq \emptyset$ are the solution subspace of

$$(\boldsymbol{a}, \boldsymbol{b}) = \arg\min_{\boldsymbol{a}\in\mathcal{A}} \max_{\boldsymbol{b}\in\mathcal{B}} f(\boldsymbol{a}, \boldsymbol{b}) = \arg\max_{\boldsymbol{b}\in\mathcal{B}} \min_{\boldsymbol{a}\in\mathcal{A}} f(\boldsymbol{a}, \boldsymbol{b}) \tag{40}$$

and then we have the following:

- if $(\boldsymbol{a}, \boldsymbol{b}) \in \bar{\mathcal{A}} \times \bar{\mathcal{B}}$, and then $(\boldsymbol{a}, \boldsymbol{b})$ is a NE.
- if $(\boldsymbol{a}, \boldsymbol{b}) \notin \bar{\mathcal{A}} \times \bar{\mathcal{B}}$, and then $(\boldsymbol{a}, \boldsymbol{b})$ could not be a NE.

Proof: Suppose $(\boldsymbol{a}_1, \boldsymbol{b}_1) \in \bar{\mathcal{A}} \times \bar{\mathcal{B}}$, $(\boldsymbol{a}_2, \boldsymbol{b}_2) \in \mathcal{A} \times \mathcal{B}$ but $(\boldsymbol{a}_2, \boldsymbol{b}_2) \notin \bar{\mathcal{A}} \times \bar{\mathcal{B}}$. It is direct to get $(\boldsymbol{a}_1, \boldsymbol{b}_2) \in \mathcal{A} \times \mathcal{B}$ and $(\boldsymbol{a}_2, \boldsymbol{b}_1) \in \mathcal{A} \times \mathcal{B}$. The proof of $(\boldsymbol{a}_1, \boldsymbol{b}_1)$ being a NE is straightforward. The following will focus on the second by contradiction. Assume $(\boldsymbol{a}_2, \boldsymbol{b}_2)$ to be a NE. With the definition of NE and the properties of $(\boldsymbol{a}_1, \boldsymbol{b}_1)$, we have

$$\begin{aligned} f(\boldsymbol{a}_2, \boldsymbol{b}_2) &\ge f(\boldsymbol{a}_2, \boldsymbol{b}_1) \ge f(\boldsymbol{a}_1, \boldsymbol{b}_1) \\ f(\boldsymbol{a}_2, \boldsymbol{b}_2) &\le f(\boldsymbol{a}_1, \boldsymbol{b}_2) \le f(\boldsymbol{a}_1, \boldsymbol{b}_1). \end{aligned} \tag{41}$$

As a result,

$$f(\boldsymbol{a}_2, \boldsymbol{b}_2) = f(\boldsymbol{a}_1, \boldsymbol{b}_1) = \min_{\boldsymbol{a}\in\mathcal{A}} \max_{\boldsymbol{b}\in\mathcal{B}} f(\boldsymbol{a}, \boldsymbol{b}) = \max_{\boldsymbol{b}\in\mathcal{B}} \min_{\boldsymbol{a}\in\mathcal{A}} f(\boldsymbol{a}, \boldsymbol{b}) \tag{42}$$

holds true, and then we obtain $(\boldsymbol{a}_2, \boldsymbol{b}_2) \in \bar{\mathcal{A}} \times \bar{\mathcal{B}}$. Clearly, the conclusion contradicts the assumption; therefore, $(\boldsymbol{a}_2, \boldsymbol{b}_2)$ could not be a NE. ■

The first result of Proposition 5 can also be found in discrete matrix game analysis [21], while the proof procedure is borrowed from the critical points analysis in [33]. Here, we restate them in Proposition 5 for mathematical completeness. Based on Proposition 5, the NEs for the MI game are summarized in the following proposition.

Proposition 6: Defining the jamming power threshold P_n as in (33), the NEs for the MI based TPZS games are the following:

B1) if $K = n_t$, the NE exists and can be obtained via (32) or (37);

B2) if $K > n_t$ and $P_b \le P_n$, the NE exists and it is the common solution of (32) and (37);

B3) if $K > n_t$ and $P_b > P_n$, the NE does not exist.

Proof: If $K = n_t$, (32) and (37) are the same. With Proposition 5, result B1) holds true. As for $K > n_t$, it is easy to find that iff $\sum_{i=1}^{K-n_t} \sigma^b_i \neq 0$, (32) and (37) could induce two different results. Equivalently, if $P_b \le P_n$, the *minimax* and *maxmin* optimizations will have the same solution; therefore, B2) holds true. The third one is somewhat complex, and it will be proven by contradiction. Let

$$\begin{aligned} \tilde{\mathbf{\Lambda}}_s &= \text{diag}\left(\left[\tilde{\sigma}^s_1, \cdots, \tilde{\sigma}^s_p, \mathbf{0}_{1\times(n_t-p)}\right]\right) \\ \tilde{\mathbf{\Lambda}}_b &= \text{diag}\left(\left[\tilde{\sigma}^b_1, \tilde{\sigma}^b_2, \cdots, \tilde{\sigma}^b_K\right]\right) = \boldsymbol{P}_1^T \mathbf{\Lambda}_b \boldsymbol{P}_1 \\ \tilde{\mathbf{\Lambda}}_w &= \text{diag}\left(\left[\tilde{\sigma}^w_1, \tilde{\sigma}^w_2, \cdots, \tilde{\sigma}^w_K\right]\right) = \boldsymbol{P}_1^T \mathbf{\Lambda}_w \boldsymbol{P}_1 \end{aligned} \tag{43}$$

where $p \le n_t$ is the number of activated signal subspaces. Assuming that $(\tilde{\mathbf{\Lambda}}_s, \tilde{\mathbf{\Lambda}}_b)$ is a NE, we must have the following:

C1) $\tilde{\sigma}^b_{p+1} = \tilde{\sigma}^b_{p+2} = \cdots = \tilde{\sigma}^b_K = 0$;
C2) $\max\{\tilde{\sigma}^w_i + \tilde{\sigma}^b_i, 1 \le i \le p\} \le \min\{\tilde{\sigma}^w_i, p+1 \le i \le K\}$;
C3) $\tilde{\sigma}^w_1 + \tilde{\sigma}^b_1 = \tilde{\sigma}^w_2 + \tilde{\sigma}^b_2 = \cdots = \tilde{\sigma}^w_p + \tilde{\sigma}^b_p$.

Note the following: 1) if $\sum_{i=p+1}^{K} \tilde{\sigma}^b_i > 0$, the target will have incentive to deviate from $(\tilde{\mathbf{\Lambda}}_s, \tilde{\mathbf{\Lambda}}_b)$, since reallocating the power portion $\sum_{i=p+1}^{K} \tilde{\sigma}^b_i$ into the p selected subspaces will result in a smaller MI; 2) if C2) is not satisfied, the radar will have incentive to deviate from $(\tilde{\mathbf{\Lambda}}_s, \tilde{\mathbf{\Lambda}}_b)$ with a similar argument; and 3) C3) can be obtained with an approach similar to that of Appendix C. Recalling the ordering inequality (15),

$$\begin{aligned}\min\{\tilde{\sigma}^w_i, p+1 \le i \le K\} &\le \min\{\sigma^w_i, 1 \le i \le K-p\}\\ &= \sigma^w_{K-p} \le \sigma^w_{K-n_t} \qquad (44)\end{aligned}$$

holds true. Based on C1), C3), and (15), we have

$$\begin{aligned}\max\{\tilde{\sigma}^w_i + \tilde{\sigma}^b_i, 1 \le i \le p\} &= P_b/p + \frac{1}{p}\sum_{i=1}^{p} \tilde{\sigma}^w_i\\ &\ge P_b/p + \frac{1}{p}\sum_{i=1}^{p} \sigma^w_{K+1-i}. \qquad (45)\end{aligned}$$

Combining (44) and (45) with C2), we get

$$\sigma^w_{K-n_t} \ge P_b/p + \frac{1}{p}\sum_{i=1}^{p} \sigma^w_{K+1-i}. \qquad (46)$$

Substituting the precondition $P_b > P_n$ into (46), we obtain

$$\begin{aligned}\sigma^w_{K-n_t} &> P_n/p + \frac{1}{p}\sum_{i=1}^{p} \sigma^w_{K+1-i}\\ &= \sigma^w_{K-n_t} + \frac{1}{p}\sum_{i=p+1}^{n_t} \underbrace{\left(\sigma^w_{K-n_t} - \sigma^w_{K+1-i}\right)}_{\ge 0}. \qquad (47)\end{aligned}$$

Apparently, the NE assumption leads to a contradiction in (47). Therefore, $(\tilde{\mathbf{\Lambda}}_s, \tilde{\mathbf{\Lambda}}_b)$ cannot be a NE, and B3) holds true. ■

The existence of a NE depends on K, n_t, P_n, and P_b. As for $K > n_t$, it may not always exist. The behavior of game players is easy to predict if the NE exists; otherwise, it will depend on other factors that are more intricate to formalize. Regarding a *matrix zero-sum game* with finite strategies, one may resort to *mixed-strategy* approach, in which each player chooses a number of strategies with a reasonable probability [21]. Interestingly, the existence of a pure- (or mixed-) strategy NE is guaranteed in theory for a matrix zero-sum game. The games in a continuous space naturally have an infinite number of pure (and mixed) strategies. In the absence of a NE, strategy analysis becomes rather difficult and heuristic. Although the game may not converge to a stationary strategy pair in this case, the players at least can play the minmax or maxmin strategy to avoid the worst case.

The minmax and maxmin solutions are unique for $K = n_t$, and hence so is the NE. As for $K > n_t$ and $P_b \le P_n$, the uniqueness of NE should be discussed case-by-case, as follows.

- If $\sigma^w_{K-n_t} \ne \sigma^w_{K-n_t+1}$, the NE is unique, since the signal subspace selection for the radar is unique, and consequently so is the maxmin optimization result.
- If $\sigma^w_{K-n_t} = \sigma^w_{K-n_t+1}$ and $P_s \le P_n - P_b$, the power allocation interaction is played in the subspaces corresponding to σ^w_i, where $\sigma^w_{K-n_t+1} < i \le K$, and the minmax (or maxmin) solution is unique, so is the NE.
- If $\sigma^w_{K-n_t} = \sigma^w_{K-n_t+1}$ and $P_s > P_n - P_b$, the NE is not unique as both maxmin and minmax optimizations have multiple solutions. Again, let $K = 3$, $n_t = 2$, $\sigma^w_1 = \sigma^w_2 = 3$, $\sigma^w_3 = 1$, $P_b = 1$, and $P_s = 3$. Both $(\mathbf{\Lambda}_b = \mathrm{diag}([0,0,1], \mathbf{\Gamma}_s = \mathrm{diag}([1,0,2]))$ and $(\mathbf{\Lambda}_b = \mathrm{diag}([0,0,1], \mathbf{\Gamma}_s = \mathrm{diag}([0,1,2]))$ are NEs. It is interesting to see that the strategy for the target is the same across all the NEs since $P_b \le P_n$. Therefore, the radar could accurately infer the target's response and guarantee their strategy pair to be a NE.

In short, the NE may not be unique under some circumstances. However, their final strategy pair may converge to a NE, as the radar could predict the power allocation of the target.

B. A Conjecture for $K > n_t$

This subsection focuses on a special case where the noise subspaces are even: $\sigma^w_i = \sigma_w$ for $\forall i$. Based on Section V, the minmax strategy of the target is to spread uniformly its power into K noise subspaces, while its power allocation strategy for the maxmin game is also uniform but limited to the n_t selected ones. The best strategy of the radar is always uniform for both games. As for the MI based TPZS games with symmetric information, two questions follow:

1) Will the MI be increased if the target uniformly concentrates its jamming power in $(K-m)$, where $0 \le m < K$, rather than K noise subspaces?
2) Will the MI be increased if the radar uniformly injects its power to n_c, where $0 < n_c \le n_t$, instead of n_t subspaces?

These questions will be next explored under the probabilistic framework. Suppose that the MIMO radar uniformly allocates its power into n_t randomly selected noise subspaces, so that the probability that exactly l jamming-free subspaces are chosen as

$$P(l) = \binom{m}{l}\binom{K-m}{n_t-l} \Big/ \binom{K}{n_t}. \qquad (48)$$

Define $\theta = \min\{n_t, m\}$ as the maximum number of the selected jamming-free subspaces, and $\beta = \max\{0, n_t-(K-m)\}$ as its minimum number; as a result, the MI expectation is formulated as

$$\bar{I}_{\mathrm{mix}}(m) = \sum_{i=\beta}^{\theta} P(i)\left[(n_t - i)g_1(m) + i g_2\right] \qquad (49)$$

where

$$g_1(m) = n_r \log\left(\frac{\sigma_h^2 P_s/n_t}{P_b/(K-m) + \sigma_w} + 1\right) \qquad (50)$$

$$g_2 = n_r \log\left(\frac{\sigma_h^2 P_s/n_t}{\sigma_w} + 1\right) \qquad (51)$$

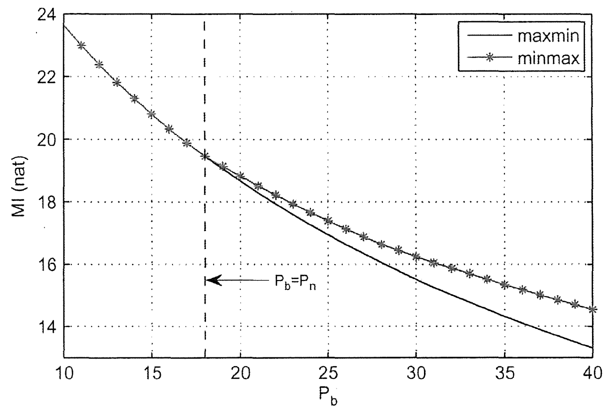

Fig. 2. The MI of at SEs as functions of P_b for the minmax and maxmin games, where $P_s = 40$. If $P_b \le P_n$, the two curves overlap, while the minmax curve is always above the maxmin one for $P_b > P_n$.

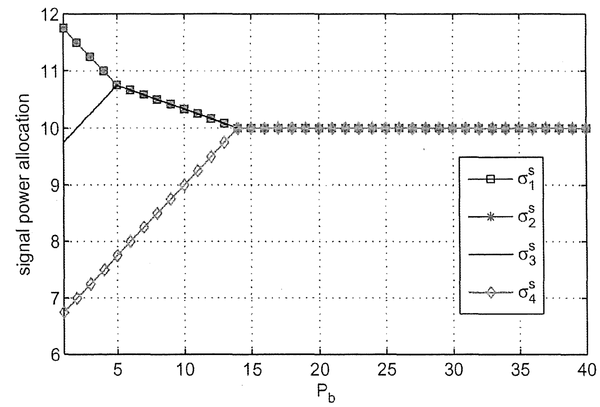

Fig. 4. The waveform power allocation strategy at SEs as functions of P_b for the minmax or maxmin games, where $P_s = 40$. With the increase of P_b, the strategy goes to uniform.

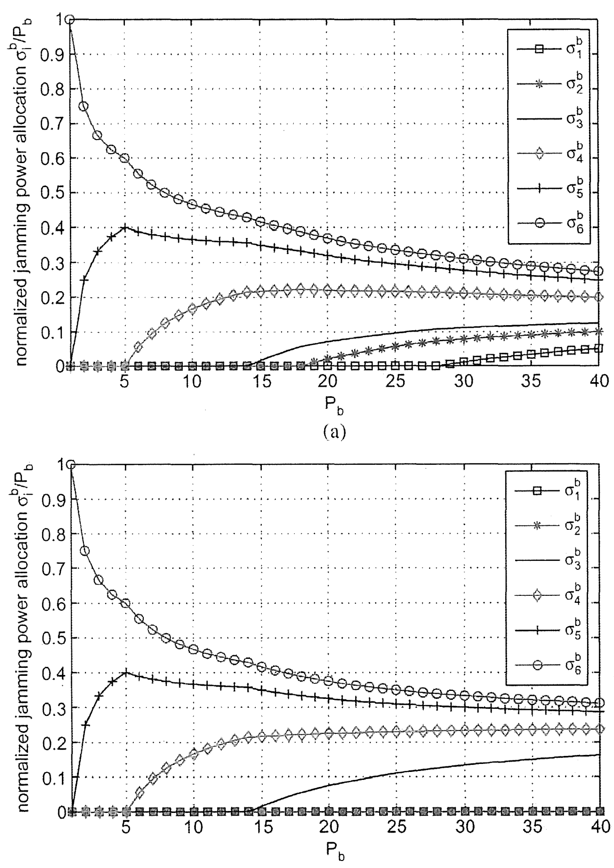

Fig. 3. The normalized jamming power allocation strategies at SEs as functions of P_b for the minmax and maxmin games, where $P_s = 40$. (a) minmax game; (b) maxmin game.

respectively represent the contribution of a jammed and a jamming-free subspace. The comparison between $\bar{I}_{\text{mix}}(m)$ and the minmax result

$$\bar{I}_{\text{minmax}} = n_r n_t \log\left(\frac{\sigma_h^2 P_s/n_t}{P_b/K + \sigma_w} + 1\right) \tag{52}$$

is mathematically intricate. Based on various computer simulation, we make the following conjecture:

Conjecture 1: Let $P_b > 0$, $P_s > 0$, and $K > n_t$. If $m > 0$, we have $\bar{I}_{\min\max} < \bar{I}_{\text{mix}}(m)$.

We will demonstrate Conjecture 1 numerically in Section VI. If Conjecture 1 holds true, the target has no incentive to shrink the size of its jamming space. Then the first question could be answered. Now, the second one becomes clear. If the target uniformly distributes its power in the entire noise space, the radar will utilize n_t subspaces and uniformly pour its power within them.

Theoretically, the NE does not exist for this special case. The noncooperative game would be unpredictable, as no single strategy pair can dominate the others. However, if the two players consider the *expectation of loss* instead of *loss* itself, the minmax solution would be the best strategy pair for both of them. In other words, the minmax solution is not the best based on equilibrium theory; however, it is the best one *in probability* when one player could not accurately infer the strategy of the other.

VII. Numerical Results

A. Examples for Hierarchical Games

This subsection concentrates on the hierarchical games. In simulations, we set $n_t = 4$, $n_r = 6$, and $K = 6$. The noise powers are respectively chosen as $\sigma_1^w = 10$, $\sigma_2^w = 8$, $\sigma_3^w = 7$, $\sigma_4^w = 4$, $\sigma_5^w = 2$, and $\sigma_6^w = 1$. Finally, $\sigma_h = 1$ for simplicity.

In the first example, P_s is fixed at 40, while P_b varies from 1 to 40. The MI of *minmax* and *maxmin* solutions for the hierarchical games are depicted in Fig. 2. Clearly, both of them are decreasing functions of P_b. Moreover, if P_b is below a certain level, $P_n = 18$, the minmax and maxmin solutions are the same, while the minmax curve is always above the maxmin one if $P_b > P_n$. This coincides with the theoretical analysis in Section V. The dashed threshold line also acts the bound for the existence of pure strategy NE for the games with symmetric information.

Figs. 3 and 4 show their power allocation equilibria. The equilibria perform like a two-step water-filling: firstly, a noise subspace with a low σ_i^w obtains more jamming power; second, the

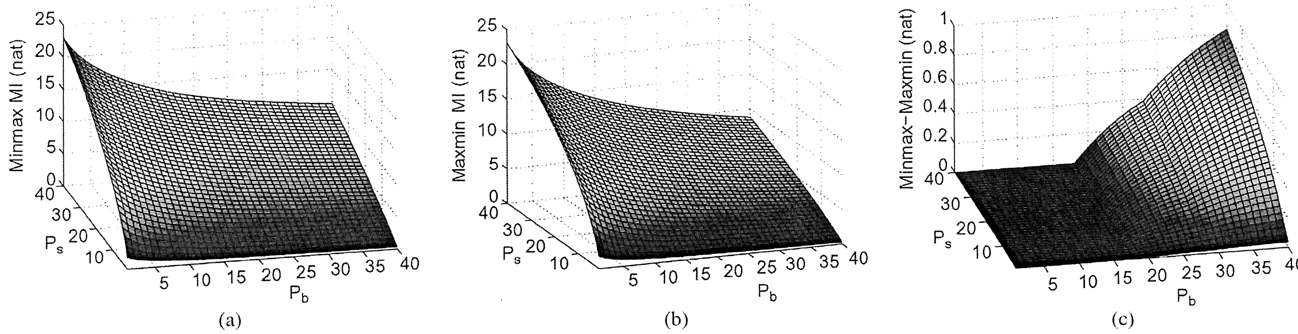

Fig. 5. The MI of the minmax and maxmin games as well as their difference at SEs as functions of P_b and P_s. (a) minmax result: $\bar{I}_{\min\max}$; (b) maxmin result: $\bar{I}_{\max\min}$; (c) $\bar{I}_{\min\max} - \bar{I}_{\max\min}$.

subspace with a small $(\sigma_i^b + \sigma_i^w)$ will obtain more waveform energy. From Fig. 3, we observe that all the six σ_i^b's will be sequentially activated with the increase of P_b for the minmax results, while only σ_j^b's, $3 \leq j \leq 6$, will be sequentially activated for the latter. From Fig. 4, we know that the waveform power allocation strategy tends toward uniform with an increase of P_b. Here is an explanation. If P_b is sufficient large, all the selected σ_i^b's will be activated in the first water-filling step; therefore, we have $\sigma_i^b + \sigma_i^w = \lambda$ for $\forall i$. As the σ_i^s's are obtained by a water-filling on $(\sigma_i^b + \sigma_i^w)$'s, the optimal power allocation strategy becomes uniform. Note that since the waveform power allocation strategies are the same for both games, only one plot is shown.

B. Examples for the Conjecture

This subsection demonstrates Conjecture 1 in Section VI. For simplicity, we set $\sigma_h = 1$ and $\sigma_w = 1$. The numerical results with different system parameters are listed in Fig. 6. In the first simulation, we set $K = 8$, $n_t = 5$, and $P_s = 40$, while the parameters are chosen as $K = 13$, $n_t = 9$, and $P_s = 20$ in the second one. In the figure, "$n_c = 4$" denotes the MI curve obtained by shrinking the size of the waveform space from n_t to n_c. From those figures, we observe that the MIs will be enlarged if the size of jamming space $(K - m)$ is reduced, and they are always above the minmax results. However, if the size of the waveform space becomes smaller, the MIs will be lower than the minmax results. This is consistent with Conjecture 1. Note that many simulations with different parameters have been performed; the observations coincide. These presented are representative of all that we have seen.

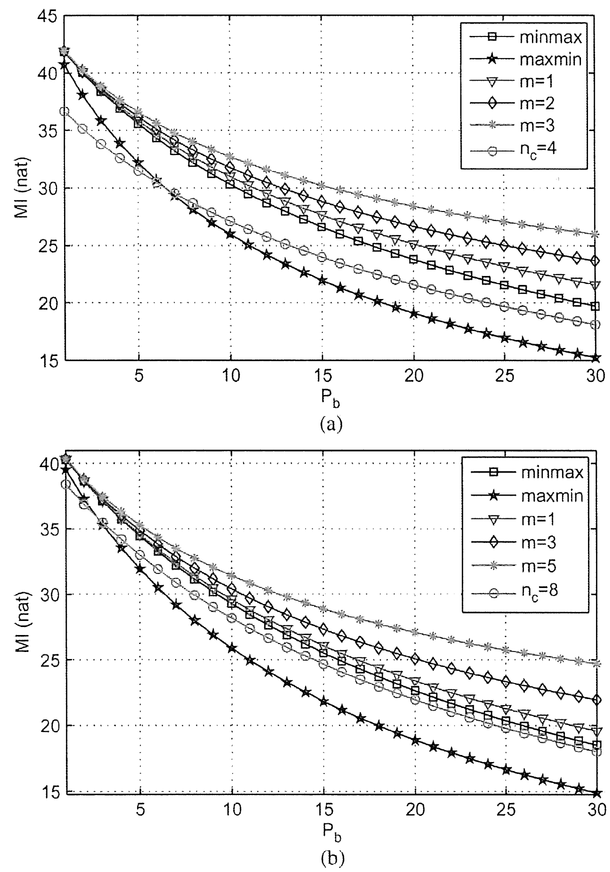

Fig. 6. Comparisons of $\bar{I}_{\min\max}$, $\bar{I}_{\max\min}$, $\bar{I}_{\mathrm{mix}}(m)$, and $\bar{I}_{\min\max}(n_c)$ for different parameter sets. If the size of jamming space $(K - m)$ decreases, the $\bar{I}_{\mathrm{mix}}(m)$ will become larger. (a) $K = 8$, $n_t = 5$, and $P_s = 40$; (b) $K = 13$, $n_t = 9$, and $P_s = 20$.

VIII. Conclusion

The interaction between a target and a MIMO radar—both smart—is investigated from a game theory perspective. Since the target and the radar are completely hostile, their interaction is modeled as a two-person zero-sum game, and we adopt mutual information as our criterion. Unilateral, hierarchical, and symmetric games are studied based on the available information set for each player. The optimal strategies for the unilateral games are forms of water-filling, and they can be analytically derived via constrained optimization techniques. Assuming conservativeness and rationality, the optimal strategies for the hierarchical games are Stackelberg equilibria, of which the closed-form expressions can be considered as two-step water-fillings. Nash equilibria are the optimal strategies for the third case; its existence conditions are discussed.

Appendix A
Several Useful Lemmas

Lemma 1: Supposing the eigendecomposition of two $n \times n$ positive semidefinite Hermitian matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ be $\boldsymbol{A} = \boldsymbol{U}_A \boldsymbol{\Lambda}_A \boldsymbol{U}_A^H$ and $\boldsymbol{B} = \boldsymbol{U}_B \boldsymbol{\Lambda}_B \boldsymbol{U}_B^H$ respectively, where $\boldsymbol{\Lambda}_A \triangleq$

$\text{diag}([\alpha_1, \alpha_2, \cdots, \alpha_n])$ with $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$, and $\boldsymbol{\Lambda}_B \triangleq \text{diag}([\beta_1, \beta_2, \cdots, \beta_n])$ with $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$, we have

$$\prod_{i=1}^{n}(\alpha_i + \beta_i) \leq \det(\boldsymbol{A} + \boldsymbol{B}) \leq \prod_{i=1}^{n}(\alpha_i + \beta_{n+1-i}). \quad (53)$$

The lower bound holds iff $\boldsymbol{U}_A = \boldsymbol{U}_B$, while the upper one is achieved iff $\boldsymbol{U}_A = \boldsymbol{U}_B\boldsymbol{P}$, where

$$\boldsymbol{P} = \begin{bmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{bmatrix} \quad (54)$$

is a skew identity matrix.

Proof: Proof can be found in [34]. ■

Lemma 2: Suppose that the diagonal elements of matrices $\boldsymbol{A} = \text{diag}([a_1, a_2, \cdots, a_n])$ and $\boldsymbol{B} = \text{diag}([b_1, b_2, \cdots, b_n])$ are both in decreasing orders: $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $b_1 \geq b_2 \geq \cdots \geq b_n > 0$, and then we have

$$\det(\boldsymbol{AB}^{-1} + \boldsymbol{I}_n) \leq \det\left(\boldsymbol{P}_1\boldsymbol{AP}_1^T\boldsymbol{B}^{-1} + \boldsymbol{I}_n\right) \leq \det(\boldsymbol{PAP}^T\boldsymbol{B}^{-1} + \boldsymbol{I}_n). \quad (55)$$

where $\boldsymbol{P}_1$ is an arbitrary permutation matrix, while $\boldsymbol{P}$ is defined in (54).

Proof: Lemma 2 is a special case of Lemma 1. ■

Lemma 3: Define $f_2(x) = \log(x)$, where $x > 0$. Let $x_1 > x_2 > 0$, and then we have

$$f_2(x_1) + f_2(x_2) < f_2(x_1 - \Delta x) + f_2(x_2 + \Delta x) \quad (56)$$

where $0 < \Delta x \leq \frac{(x_1+x_2)}{2}$.

Proof: Since $f_2(x)$ is differentiable, we have

$$\begin{aligned} c_1 &\triangleq f_2(x_1 - \Delta x) + f_2(x_2 + \Delta x) - f_2(x_1) - f_2(x_2) \\ &= \int_{x_2}^{x_2+\Delta x} \frac{\partial f_2(x)}{\partial x}dx - \int_{x_1-\Delta x}^{x_1} \frac{\partial f_2(x)}{\partial x}dx \\ &= \int_{x_2}^{x_2+\Delta x} \frac{1}{x}dx - \int_{x_1-\Delta x}^{x_1} \frac{1}{x}dx. \end{aligned} \quad (57)$$

Since $\frac{1}{x}$ is monotonic decreasing, and since $x_1 > x_1 - \Delta x \geq x_2 + \Delta x > x_2$, we have

$$c_1 > \int_{x_1-\Delta x}^{x_1} \frac{1}{x}dx - \int_{x_1-\Delta x}^{x_1} \frac{1}{x}dx = 0. \quad (58)$$

Therefore, the result can be proven. ■

Lemma 4: Let $\mathcal{X}$ be a compact convex subset of a linear space and $\mathcal{Y}$ be a convex subset of a linear space. If $f(\boldsymbol{x}, \boldsymbol{y})$ is a real-valued function on $\mathcal{X} \times \mathcal{Y}$ with the following:

- $f(\boldsymbol{x}, \cdot)$ is upper semicontinuous and *quasiconcave* on $\mathcal{Y}$, for $\forall \boldsymbol{x} \in \mathcal{X}$;
- $f(\cdot, \boldsymbol{y})$ is lower semicontinuous and *quasiconvex* on $\mathcal{X}$, for $\forall \boldsymbol{y} \in \mathcal{Y}$;

and then we have

$$\min_{\boldsymbol{x}\in\mathcal{X}} \max_{\boldsymbol{y}\in\mathcal{Y}} f(\boldsymbol{x}, \boldsymbol{y}) = \max_{\boldsymbol{y}\in\mathcal{Y}} \min_{\boldsymbol{x}\in\mathcal{X}} f(\boldsymbol{x}, \boldsymbol{y}). \quad (59)$$

Proof: Lemma 4 is known as Sion's minimax theorem, and proof can be found in [32]. ■

APPENDIX B
PROOF OF PROPOSITION 1

Proof: Firstly, let $\boldsymbol{U}_b = \boldsymbol{U}_w\boldsymbol{P}_2$, and then we have

$$\begin{aligned} \max_{\boldsymbol{S}} I_b &\Longleftrightarrow \max_{\boldsymbol{S}} \det\left(\sigma_h^2\boldsymbol{SS}^H + \boldsymbol{R}_b + \boldsymbol{R}_w\right) \\ &\Longleftrightarrow \max_{\boldsymbol{S}} \det\left(\sigma_h^2\boldsymbol{SS}^H + \boldsymbol{U}_w\left(\boldsymbol{P}_1\boldsymbol{\Lambda}_b\boldsymbol{P}_1^T + \boldsymbol{\Lambda}_w\right)\boldsymbol{U}_w^H\right). \end{aligned} \quad (60)$$

Based on Lemma 1 in Appendix A, the maximum can be arrived iff $\boldsymbol{SS}^H$ shares the same eigenvectors as $\boldsymbol{R}_w$, which means $\boldsymbol{U}_s = \boldsymbol{U}_w\boldsymbol{P}_1$. Therefore, if $\boldsymbol{U}_b$ is set as $\boldsymbol{U}_w\boldsymbol{P}_2$, $\boldsymbol{U}_s$ has no incentive to deviate from $\boldsymbol{U}_w\boldsymbol{P}_1$. With a similar approach, it is easy to show that if $\boldsymbol{U}_s = \boldsymbol{U}_w\boldsymbol{P}_1$, $\boldsymbol{U}_b$ has no incentive to deviate from $\boldsymbol{U}_w\boldsymbol{P}_2$. Hence, the first conclusion can be proven.

In addition, the space equilibrium, say $(\boldsymbol{U}_w\boldsymbol{P}_1, \boldsymbol{U}_w\boldsymbol{P}_2)$, holds for arbitrary $\boldsymbol{\Lambda}_b$ and $\boldsymbol{\Gamma}_s$ values. Therefore, it is invariant to $\boldsymbol{\Lambda}_b$ and $\boldsymbol{\Gamma}_s$. In the following, we will prove its uniqueness of invariance by contradiction. Suppose that $\boldsymbol{U}_s = \bar{\boldsymbol{U}}_s \neq \boldsymbol{U}_w\boldsymbol{P}_1$ and $\boldsymbol{U}_b = \bar{\boldsymbol{U}}_b \neq \boldsymbol{U}_w\boldsymbol{P}_2$ compose another space equilibrium, which is invariant to $\boldsymbol{\Lambda}_b$ and $\boldsymbol{\Gamma}_s$. Based on Lemma 1, $\boldsymbol{SS}^H$ and $(\boldsymbol{R}_b + \boldsymbol{R}_w)$ must be simultaneously diagonalized

$$\bar{\boldsymbol{U}}_s^H\left(\bar{\boldsymbol{U}}_b\boldsymbol{\Lambda}_b\bar{\boldsymbol{U}}_b^H + \boldsymbol{U}_w\boldsymbol{\Lambda}_w\boldsymbol{U}_w^H\right)\bar{\boldsymbol{U}}_s = \boldsymbol{\Sigma}_1 \quad (61)$$

where $\boldsymbol{\Sigma}_1$ is a positive semidefinite diagonal matrix. Let $\bar{\boldsymbol{\Lambda}}_b$ be an arbitrary positive semidefinite diagonal matrix satisfying $\text{Tr}(\bar{\boldsymbol{\Lambda}}_b) = \text{Tr}(\boldsymbol{\Lambda}_b)$ and $\bar{\boldsymbol{\Lambda}}_b \neq \boldsymbol{\Lambda}_b$. As $(\bar{\boldsymbol{U}}_s, \bar{\boldsymbol{U}}_b)$ is invariant to $\boldsymbol{\Lambda}_b$, we also have

$$\bar{\boldsymbol{U}}_s^H\left(\bar{\boldsymbol{U}}_b\bar{\boldsymbol{\Lambda}}_b\bar{\boldsymbol{U}}_b^H + \boldsymbol{U}_w\boldsymbol{\Lambda}_w\boldsymbol{U}_w^H\right)\bar{\boldsymbol{U}}_s = \boldsymbol{\Sigma}_2 \quad (62)$$

where $\boldsymbol{\Sigma}_2$ shares the same properties as $\boldsymbol{\Sigma}_1$. Combining (61) and (64), we obtain that

$$\bar{\boldsymbol{U}}_s^H\bar{\boldsymbol{U}}_b(\boldsymbol{\Lambda}_b - \bar{\boldsymbol{\Lambda}}_b)\bar{\boldsymbol{U}}_b^H\bar{\boldsymbol{U}}_s = \boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2 \quad (63)$$

holds for arbitrary $\bar{\boldsymbol{\Lambda}}_b$. Since $(\boldsymbol{\Lambda}_b - \bar{\boldsymbol{\Lambda}}_b)$ and $(\boldsymbol{\Sigma}_1 - \boldsymbol{\Sigma}_2)$ are both nonzero diagonal matrices, we have $\bar{\boldsymbol{U}}_s = \bar{\boldsymbol{U}}_b\boldsymbol{P}_3$, where $\boldsymbol{P}_3$ is an arbitrary permutation matrix. Substitute it into (64), we obtain

$$\bar{\boldsymbol{U}}_b^H\boldsymbol{U}_w\boldsymbol{\Lambda}_w\boldsymbol{U}_w^H\bar{\boldsymbol{U}}_b = \boldsymbol{P}_1\boldsymbol{\Sigma}_2\boldsymbol{P}_1^T - \bar{\boldsymbol{\Lambda}}_b. \quad (64)$$

Since $(\boldsymbol{P}_1\boldsymbol{\Sigma}_2\boldsymbol{P}_1^T - \bar{\boldsymbol{\Lambda}}_b)$ and $\boldsymbol{\Lambda}_w$ are both diagonal, we get $\bar{\boldsymbol{U}}_b = \boldsymbol{U}_w\boldsymbol{P}_4$ again, where $\boldsymbol{P}_4$ is a permutation matrix too. Apparently, this contradicts the assumption, and $(\bar{\boldsymbol{U}}_s, \bar{\boldsymbol{U}}_b)$ could not be an invariant equilibrium. Therefore, $(\boldsymbol{U}_w\boldsymbol{P}_1, \boldsymbol{U}_w\boldsymbol{P}_2)$ is a unique invariant span, and Proposition 1 can be proven. ■

APPENDIX C
PROOF OF PROPOSITION 2

Proof: As opposed to (19) and (22), the objective function of (31) is not separable as λ_1 may depend on σ_i^b's. The direct application of a Lagrange multiplier optimization may not be proper. Here, we demonstrate the proposition with the help of the characteristics of optimal solutions:

D1) If $\sigma_i^s = 0$, we have $\sigma_{K+1-i}^b = 0$; if $\sigma_{K+1-i}^b \neq 0$, where $i \leq n_t$, we have $\sigma_i^s \neq 0$.

D2) For $1 \leq i < j \leq n_t$, if $\sigma_{K+1-j}^b > 0$, we have $\sigma_{K+1-i}^b > 0$.

D3) If $\sigma_{K+1-i}^b > 0$ and $\sigma_{K+1-j}^b > 0$, and then we have $\sigma_{K+1-i}^b + \sigma_{K+1-i}^w = \sigma_{K+1-j}^b + \sigma_{K+1-j}^w$; if $\sigma_{K+1-i}^b > 0$ and $\sigma_{K+1-j}^b = 0$, we have $\sigma_{K+1-i}^b + \sigma_{K+1-i}^w \leq \sigma_{K+1-j}^w$.

Results D1) and D2) can be easily verified by contradiction. Moreover, D2) is explained as that the noise subspace corresponding to a lower σ_k^w value will be earlier to receive jamming power. The following focuses on the proof of the third one. Let $\sigma_{K+1-i}^b > 0$, $\sigma_{K+1-j}^b > 0$, and $i, j \leq n_t$, and then we have $\sigma_i^s > 0$ and $\sigma_j^s > 0$ based on D1), or equivalently,

$$\lambda_1 - \frac{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w}{\sigma_h^2} > 0 \tag{65}$$

$$\lambda_1 - \frac{\sigma_{K+1-j}^b + \sigma_{K+1-j}^w}{\sigma_h^2} > 0 \tag{66}$$

$$\log \lambda_1 - \log \frac{\sigma_{K+1-i}^b + \sigma_{K+1-i}^w}{\sigma_h^2} > 0 \tag{67}$$

$$\log \lambda_1 - \log \frac{\sigma_{K+1-j}^b + \sigma_{K+1-j}^w}{\sigma_h^2} > 0. \tag{68}$$

Therefore, the contribution of σ_{K+1-i}^b and σ_{K+1-j}^b in the objective function of (31) is

$$c_1\left(\sigma_{K+1-i}^b, \sigma_{K+1-j}^b\right) = 2\log\left(\sigma_h^2 \lambda_1\right) - \log\left(\sigma_{K+1-i}^b + \sigma_{K+1-i}^w\right) - \log\left(\sigma_{K+1-j}^b + \sigma_{K+1-j}^w\right). \tag{69}$$

Without losing generality, we assume that the optimal solutions σ_{K+1-i}^b and σ_{K+1-j}^b could be

$$\sigma_{K+1-i}^b + \sigma_{K+1-i}^w > \sigma_{K+1-j}^b + \sigma_{K+1-j}^w. \tag{70}$$

Based on Lemma 3 in Appendix A, it is always possible to find a positive number Δ, where $0 < \Delta \leq \min\left\{\frac{(\sigma_{K+1-i}^b + \sigma_{K+1-i}^w - \sigma_{K+1-j}^b - \sigma_{K+1-j}^w)}{2}, \sigma_{K+1-i}^b\right\}$, satisfying

$$c_1\left(\sigma_{K+1-i}^b - \Delta, \sigma_{K+1-j}^b + \Delta\right) < c_1\left(\sigma_{K+1-i}^b, \sigma_{K+1-j}^b\right). \tag{71}$$

Therefore, σ_{K+1-i}^b and σ_{K+1-j}^b cannot be the components of an optimal solution; this contradicts the assumption. Note that in the derivation of (71), we use the fact that λ_1 will keep constant in the adding-subtracting processes, since σ_{K+1-i}^b and σ_{K+1-j}^b are divided by the same denominator σ_h^2 in waveform power constraint. The latter part of D3) can be similarly proven, and D3) holds true. D1)–D3) guarantee a water-filling solution of σ_k^b's with a total jamming power constraint. Substituting them into (20), the proposition can be proven. ■

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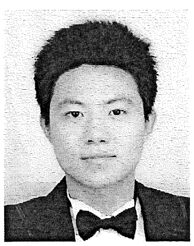

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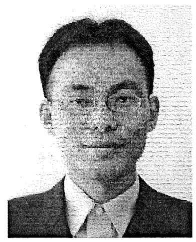

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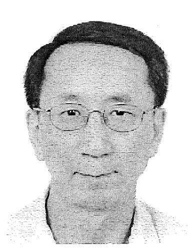

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Prof. Luh is the Vice-President for Publication Activities for the IEEE Robotics and Automation Society. He is an Associate Editor of Institute of Industrial Engineers Transactions on Design and Manufacturing, Discrete Event Dynamic Systems, and the Acta Automatica Sinica. He was the Founding Editor-in-Chief of the *IEEE Transactions on Automation Science and Engineering* from 2003 to 2007, and the Editor-in-Chief of the IEEE TRANSACTIONS ON ROBOTICS AND AUTOMATION from 1999 to 2003.