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Opportunistic Lagrangian Relaxation for Joint Replacement Policy

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Abstract: Joint replacement of multiple parts is an optimization problem where the total cost is to be minimized by coordinating the timing of replacing various parts to share resources or setup costs. Searching for a good policy for such a multi-stage combinatorial optimization problem with uncertainty could be prohibitively complex. The method developed in this paper provides a solution to a joint replacement problem of engine parts. By utilizing the characteristics of the stochastic coupling constraints, a decomposable model and the corresponding opportunistic Lagrangian relaxation (OLR) method are developed. Numerical testing shows that OLR outperforms two prevalent rule-based methods which rely on priori knowledge of the problem.

Keywords: Joint replacement, policy optimization, stochastic coupling constraints, opportunistic Lagrangian relaxation (OLR)

In the domain of maintenance strategies, joint replacement of parts is an important class of policy optimization problems for reducing costs through resource sharing. Those costs have an important impact on the economy of the equipment maintenance[1]. Nowadays, equipment maintenance service contributes more profits to the manufacturer than the equipment itself. In addition, joint replacement policy of aircraft engines or power equipment is also highly related to safety issues. Due to its economical and reliability impact, increasing attention[2−3] has been drawn to the optimization of equipment maintenance service policy.

Existing maintenance policies belong to one of two categories: corrective maintenance (CM) in which the equipment are only maintained on failures; and preventive maintenance (PM) in which periodical maintenance are scheduled in advance to avoid serious failures[4−5].

For economic and operational reasons, the two major types of policies are gradually moving toward integration, and a wide range of opportunistic maintenance (OM) policies are developed and popularized in practice[6].

Opportunistic maintenance can follow certain maintenance specifications (such as a fixed time interval), and add or omit maintenance operations in accordance with equipment status at each maintenance visit, or allow to replace some parts in case of random failures before planned replacement. This flexibility allows the maintenance decision-making to take advantage of new information to coordinate multiple parts[7−8]. It has been proved that it lowers the system failure rate and improves the system availability[9]. In [10], the steady-state strategy of a two-part problem is solved by Markov decision process (MDP), and the sensitivity analysis confirmed that the strategy economics are enhanced by introducing OM into PM. Research on OM is currently mainly done on rule-based steady-state policies, such as of threshold type or interval type[3, 10−12]. There is a lack of research on an important class of practical problems: the optimization of non-static policies under finite time horizon. Limited results exist for cases with very small number of parts.

Joint replacement studied in this paper belongs to the optimization of an OM policy that is non-static, with finite time horizon and multiple parts. The total cost in a contract duration is to be reduced by optimizing the maintenance time and the number of parts that are replaced at each maintenance. There are two types of parts in a jet engine, life-limited parts (LLP) and non-life-limited parts (non-LLP). In traditional PM policies, LLPs are forced to be replaced when their lives expire, which means their time on-wing have summed up to a certain level. On expiration of any LLP, the whole engine has to be sent to certain maintenance sites for dismantling. Without any LLP expiration, the engine may also fail, including non-LLP malfunction or other accidents. In these cases too, the engine has to be sent to maintenance site. During the maintenance, a spare engine is required or the aircraft is grounded. Besides the costs for replacing LLPs, there are also other costs including the fees for engine transportation, inspection and costs for spare engines or loss incurred by the grounding of the aircraft. It can reduce the total maintenance cost within the contract duration to replace some unexpired LLPs at maintenance caused by random failures or other expired LLPs. There are similar problems in the field of supply chain management, production scheduling, for example, joint replenishment in inventory management and maintenance of serial machines.

The optimization difficulty is threefold: non-static, combinatorial and stochastic. In addition, it is proved that the optimal policy is non-regular[2]. Since replacement decisions for different parts are made jointly at a common maintenance visit, the decisions are over the combinatorial

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part state at the maintenance visit. At each maintenance visit, the total number of possibilities of the combinatorial part state is prohibitively large to make decision over. It is known as the difficulty of combinatorial explosion . The presence of random failures further increases the difficulty to optimize the OM timing of joint part replacement.

A prevalent type of rule-based approach, threshold-type policy turns combinatorial decision-making into optimization of thresholds of part residual lives. In [3], for all parts at all time stages, an optimal threshold (OT) is solved by nero-dynamic programming (NDP) for joint replacement of jet engine parts. The uniqueness of the threshold restricts the algorithm's performance. Without structural requirements on the solutions, the solution space is intangibly large to optimize all decisions by simulation for the whole time horizon. To avoid optimizing all decisions, the one-stage analysis (OSA) method developed in [2] optimizes the average cost for each single stage with a steady-state assumption. In that paper, the author sorts the parts according to their current remaining lives at each maintenance, and then searches for an optimal threshold by simulation. The policy obtained by the above methods is limited to steady-state policy or for small number of parts.

The drawback of optimization without rules or steady state approximation is the lack of scalability to large-scale problems. In view of the uncertainty in this problem, the state variables of different parts are over the same underlying sample space. This prevents the application of methods such as Lagrangian relaxation to decouple the problem as it does with their deterministic counterparts. In [13−15], the problem can be decoupled and solved for each individual scenario (a sample path among all possibilities). When the number of scenarios is tractable, it overcomes the above difficulty of coupled state variables. In this paper, the number of scenarios is enormous and no typical representative scenarios can be used as an approximation. Without a reduction in the number of scenarios, enormous number of multipliers have to be optimized and it leads to high algorithm complexity. This complexity prevents the problem from being solved by scenario based methods.

In this paper, opportunistic Lagrangian relaxation (OLR) is correspondingly developed for this case. Corresponding to the large number of scenarios, the number of samples of the stochastic coupling constraints grows exponentially with the problem scale. Traditional LR fails due to the exponentially increased samples. This newly developed method is independent from the sample space and decouples the problem by utilizing the characteristics of the coupling constrained that identified herein. The number of multipliers is of the same order as the number of stochastic constraints, rather than the number of their samples.

Results shows OLR obtains better solutions than rule-based methods. The timing for maintenance is decided by the stochastic decision variables, rather than by a preset threshold or periodic rules. Upon the decisions for maintenance timing, the correlated part replacement decisions are decoupled and optimized from each individual part's respect, then the group of parts to be replaced are iteratively coordinated through updating the multipliers. In view that the total cost of maintenance of large equipment is very high and enough planning time is allowed, the algorithm has tight requirement on performance but less on time. OLR continuously optimizes the policy within the allowed time, and it can break the performance bottleneck of rule-based methods. Compared to MDP or stochastic programming based methods, OLR overcomes the difficulty that the policy scale increases exponentially with the number of parts and the length of the contract duration.

The rest of paper is organized as follow: Section 1 provides the problem formulation and an analysis of its stochastic characteristics and separability. In Section 2, the linear coupling constraints with only 0-1 variables are approximated by their expectation and the corresponding OLR algorithm is demonstrated. In Section 3, numerical results verify the effectiveness of OLR with increasing uncertainties. OLR decouples the decision variables and simplifies the impact of randomness, so that the difficulty in optimizing the policy is reduced and solutions are improved.

1 Engine part replacement

Table 1 is the list of notions for ease of the following derivation presentation.

Table 1 Symbol lists

Symbol	Description
T	Contract duration
N	Number of parts
S_n	Life limit of part n
$x_{n,t}$	Residual life of part n
r_t	Failure rate at t
ξ_t	Indicator of random failure at t
Δ_t	0-1 maintenance visit decision at t
$d_{n,t}$	Part replacement decision variable for part n at t
c^s	Setup cost for each maintenance visit
c^p_n	Cost for replacing part n
$\lambda^k_{n,t}$	Value of multiplier at iteration k to relax coupling constraint n
$\lambda^*_{n,t}$	The optimal value of multipliers to relax coupling constraint n
s^k	Step size for updating multipliers at iteration k
g	Sub-gradient

1.1 Model for joint replacement problem

For an engine maintenance problem with contract duration T, decisions for maintenance visits or part replacement are to be made at each stage $t = 0, \cdots, T-1$. An engine consists of N LLPs, whose life limits are S_n, $n = 1, \cdots, N$. The residual lives of LLPs decrease by one day for each day if the engine is on-wing, and remain unchanged if the engine is in maintenance. Before an LLP's residual life drops to zero, the engine has to be sent to maintenance to replace it. The maintenance caused by malfunction of non-LLPs or accidents are modeled together as random failures, and the failure rates are given as r_t, $t = 0, \cdots, T-1$ for each t. There is a fixed setup cost c^s for each maintenance visit either due to LLP expiration or random failures. There is also a replacement cost c_n^p for replacing part n. To optimize the engine maintenance visits and part replacement decisions so as to minimize the total maintenance cost, the problem is formulated into the model below with system dynamic equations and constraints.

1.1.1 Dynamic equation

The residual life of part n is $x_{n,t}$ at t, and its residual life at the beginning of the contract is known as $x_{n,0}$. The residual life $x_{n,t}$ is decreased by one for each t on-wing, and does not change for each t at maintenance. When the engine is at maintenance at t, the maintenance visit decision variable Δ_t takes value 1, and 0 otherwise. If part n is replaced at maintenance, its residual life is restored to "as new" S_n; otherwise, it remains unchanged. The dynamic equation of an LLP's residual life is therefore written as

$$x_{n,t+1} = x_{n,t} - 1 + \Delta_t + d_{n,t}(S_n - x_{n,t}), \ \forall n, t, \qquad (1)$$

where $d_{n,t}$ is a 0-1 decision variable, with value 1 for replacing part n at t, and 0 otherwise.

From the equation above, at maintenance, the residual life $x_{n,t}$ does not decrease with t. The minimal maintenance duration is one time unit. If the maintenance takes longer, additional constraints are required on Δ_t, or a modification on the contract duration in the model[16]. We do not consider these cases in this paper for simplification.

There is a hidden assumption that the engine is either on-wing or at maintenance. In practice, the idle cost of an engine is very high, so this assumption is true most of time[17].

1.1.2 Constraints

An LLP with zero residual life must be replaced, as described by the following replacement constraints

$$1 - x_{n,t} - d_{n,t} \leq 0, \ \forall n, t. \qquad (2)$$

At the end of the contract duration, i.e., at $t = T$, the residual life of part n is required to be above a given level $\boldsymbol{x}^{\mathrm{T}}$. This constraint on terminal condition is:

$$x_{n,T} \geq x_{n'}^{\mathrm{T}}, \ \forall n. \qquad (3)$$

The random failures of the engine is indicated by 0-1 variable ξ_t, which takes value 1 on failures, and zero otherwise. The distribution is given by the failure rate

$$\Pr(\xi_t) = r_t I_{\{\xi_t=1\}} + (1 - r_t) I_{\{\xi_t=0\}}, \qquad (4)$$

where I_A is the indicator function. For simplification, the failure rate is assumed to be independent of time and state of LLP. It is noted that this assumption is not a prerequisite for the model and algorithm in this paper.

When any LLP expires or random failures take places, a maintenance visit is requisite. It means that when the replacement decision $d_{n,t}$ or failure indicator ξ_t takes the value of one, Δ_t must be one also. These two constraints are written as

$$d_{n,t} - \Delta_t \leq 0, \quad \text{(a.e.)} \quad \forall n, t, \qquad (5)$$

and

$$\xi_t - \Delta_t \leq 0, \quad \text{(a.e.)} \quad \forall n, t. \qquad (6)$$

Because of random failures, the replacement decision $d_{n,t}$ and maintenance decision Δ_t are all stochastic variables. From the system dynamic equation (1), it can be seen that the LLP residual lives are also stochastic variables. The above constraints (5) and (6) should be satisfied in all conditions, which is one of the major difficulties to be overcome.

1.1.3 Objective function

The maintenance costs consist of two parts: the setup cost c^s is a fixed, global cost that includes the transportation cost, maintenance supplies, labor, repairs and inspection; the other cost is the LLP replacement cost for each replacement part, including the new LLPs' costs, installation and labor etc. The cost for each time unit is given as

$$C_t\left(\mathbf{x}_t, d_t, \Delta_t\right) = c^s \Delta_t + \sum_n c_n^p d_{n,t}. \qquad (7)$$

In this optimization problem, the decision variables are maintenance decision Δ_t and LLP replacement decision $d_{1,t}, \cdots, d_{N,t}$. The total cost to minimize is for all T time units, so the objective function is given as

$$\min_{\{d_t, \Delta_t\}} J_0, \quad \text{with } J_0 \equiv \mathrm{E}\left[\sum_{t=0}^{T-1} C_t(\mathbf{x}_t, d_t, \Delta_t)\right]. \qquad (8)$$

Because the state variables and decision variables are stochastic, the objective is the expectation of the total cost. The expectation is taken over the random failures during the whole contract duration $[0, T-1]$, that is, over the failure sequence $\{\xi_0, \cdots, \xi_{T-1}\}$.

1.2 Model analysis

In the formulation above, the objective function is additive in the decision variables. In addition, among all constraints, some constraints include only replacement decisions of a single LLP $d_{n,t}$, for example, constraints (2) and (3); or include only maintenance decision Δ_t, such as (6). In constraint (5), both replacement decision $d_{n,t}$ and engine maintenance decision Δ_t are involved, so constraint (5) is called the coupling constraint. It can be seen that the coupling constraint (5) is linear with respect to the decision variables; on the other hand, the objective (8) are summation of all decision variables. In deterministic problems, linear coupling constraints and additive objective function are the necessary and sufficient conditions of the separability of the formulation.

After relaxing the coupling constraints through the use of multipliers, the Lagrangian function generated consists of two parts, the original function and the penalty term. Both of them are linear combinations of decision variables, therefore the Lagrangian problem can be decomposed into several individual subproblems. The maintenance decision Δ_t in system dynamics equation (1) is regarded as given during the solving process, so (1) is regarded as the constraint for a single LLP's replacement decision $d_{n,t}$. The potential deviation resulting from considering Δ_t as given is compensated by the algorithm, as shown in the next section.

At t, the previous random failure sequence, i.e., $\xi^{0,t-1} = \{\xi_0, \cdots, \xi_t\}$, leads to stochastic residual lives' combinations. The LLP state combination, $(x_{1,t}, \cdots, x_{N,t})$, requires different decisions to be made. Thus, the stochastic coupling constraints at t is equivalent to 2^t deterministic realizations of this coupling constraint, which leads to the key difficulty of this decision problem.

2 Approximate model and OLR algorithm

2.1 Approximate model

To overcome the difficulty mentioned above, an expectation is used as a substitute for the left-hand side of the stochastic coupling constraint (5), which gives the approximate coupling constraint:

$$\mathrm{E}(d_{n,t} - \Delta_t) \le 0, \quad \forall n, t. \tag{9}$$

This new coupling constraint is a deterministic inequality, and it is a necessary condition for the original stochastic coupling constraints to be satisfied. By denoting the left-hand side of (9) as $\bar{g}_{n,t}$, because the two decision variables are both 0-1 variables, one concludes that

$$\begin{aligned}\bar{g}_{n,t} = &\Pr(d_{n,t} = 1) - \Pr(\Delta_t = 1) = \\ &\sum_{\omega} \Pr(d_{n,t}(\omega) = 1) \Pr(\Delta_t(\omega) = 0) - \\ &\sum_{\omega} \Pr(d_{n,t}(\omega) = 0) \Pr(\Delta_t(\omega) = 1).\end{aligned} \tag{10}$$

The value of $\bar{g}_{n,t}$ represents the difference between the probability that the original coupling constraint is violated and the probability that it is non-active. It is noted that minimizing the value of $\bar{g}_{n,t}$ is equivalent to satisfying the original coupling constraints. We thus arrrive at an approximate problem with an objective consistent with the original optimization problem, which is defined by the original objective (8), constraints (1) ~ (4), (6) and (9). This new problem is a stochastic optimization with deterministic coupling constraints. If we use multipliers $\{\lambda_{n,t} \ge 0\}$ to relax the corresponding constraint (9), the Lagrangian function is:

$$\begin{aligned}L(\Delta_t, d_{n,t}, \lambda_{n,t}) = &\sum_t \left[c^s - \sum_n \lambda_{n,t}\right] \mathrm{E}\Delta_t + \\ &\sum_{n,t} (c_n^p + \lambda_{n,t})\, \mathrm{E}d_{n,t}.\end{aligned} \tag{11}$$

This Lagrangian function can be decomposed into N LLP subproblems and each engine maintenance subproblem that can be minimized individually as follow.

2.2 LLP subproblem

The objective function of the LLP subproblem for part n is

$$\begin{aligned}L_n(\lambda) = &\mathrm{E} \sum_{t=0}^{T-1} (c_n^p + \lambda_{n,t}) d_{n,t} = \\ &\sum_{t=0}^{T-1} (c_n^p + \lambda_{n,t}) \mathrm{E}d_{n,t}, \quad n = 1, \cdots, N,\end{aligned} \tag{12}$$

with constraints (1) ~ (3). Part n's state transition equation involves maintenance decision variable Δ_t, which represents the maintenance duration. During this period of time, the state of the LLP remains unchanged. The maintenance variable Δ_t is an uncertain decision variable, and should be solved through its individual subproblem. Its existence in the LLP subproblem affects the solution process for replacement decisions. As an approximation, an alternative state transition equation is used in LLP subproblems:

$$\begin{aligned}x_{n,t+1} = &x_{n,t} - 1 + \max(d_{n,t}, \Delta_t) + \\ &d_{n,t}(S_n - x_{n,t}), \quad \forall n, t.\end{aligned} \tag{13}$$

The reason for using $\max(d_{n,t}, \Delta_t)$ instead of Δ_t above is as follows: after decomposition, there is no guarantee that the solution obtained in separate subproblems for decisions $d_{n,t}$ and Δ_t can satisfy the original constraint (5), so it may lead to an incorrect state transition dynamics. The alternative term $\max(d_{n,t}, \Delta_t)$ can ensure that an LLP's residual life is reasonable at expiration, when $d_{n,t} = 1$, whether the current solution Δ_t is feasible or not. In addition, when an LLP is not replaced but the engine is at maintenance, this alternative term also ensures that this LLP's state remains unchanged.

It can be seen from (12) that L_n is separable in t; the minimization on L_n can therefore be solved by dynamic programming with state transition equation (13), replacement constraints (2) and terminal condition constraints (3). It is worth noting that the alternative LLP subproblems can be solved by deterministic dynamic programming, the reason is as follows:

1) In (12), because the cost $c_n^p + \lambda_n$ for each stage is deterministic, the expectation symbol permutes with the the summation and multiplication symbols. To optimize the objective function, only decisions for $\mathrm{E}d_{n,t}$ are required rather than for $d_{n,t}$.

2) It can be seen from (6) that the impact of uncertainty is only on the maintenance decision variable Δ_t. In the alternative state transition (13), the existence of Δ_t introduces uncertainty and have impact on the LLP's state transition probability. According to Bellman's optimality principal, there exists a deterministic solution that minimizes the expected cost-to-go. To solve the LLP subproblem, thus, there is no need to worry about a stochastic solution for decision $d_{n,t}$.

3) As described above, the decision variables are deterministic, that is, $\mathrm{E}d_{n,t} = d_{n,t}$ in LLP subproblem. In the text hereafter, the term "decision variable" does not distinguish between $\mathrm{E}d_{n,t}$ and $d_{n,t}$.

2.3 Maintenance visit subproblem

The objective function of the maintenance visit subproblem is:

$$L_a(\Delta_t) = \mathrm{E}\sum_t \left[c^s - \sum_n \lambda_{n,t}\right]\Delta_t = \sum_t \left[c^s - \sum_n \lambda_{n,t}\right]\mathrm{E}\Delta_t. \quad (14)$$

Note that the objective function is also linear and additive with stage t, therefore its optimal solution under constraint (6) is:

$$\mathrm{E}\Delta_t = \begin{cases} r_t, & c^s - \sum\limits_n \lambda_{n,t} > 0, \\ 1, & c^s - \sum\limits_n \lambda_{n,t} < 0, \\ \text{Any} \in [0, r_t], & c^s - \sum\limits_n \lambda_{n,t} = 0. \end{cases} \quad (15)$$

This explicit solution implies the following facts: starting from the solution with zero $\lambda_{n,t}$ and no maintenance, the penalty is increased when $g_{n,t} = d_{n,t} - \Delta_t > 0$ (i.e., increase $\lambda_{n,t}$), when the penalty is high enough for constraints violation, i.e., the sum of multipliers is more than the maintenance setup cost, the optimal maintenance decision is 1 for this stage t, otherwise it remains 0.

Notice that this explicit solution is linear for the multipliers and quite sensitive to their values near the critical value. In order to reduce the oscillation, we use Bang-bang control in the iterative solution process. Near the critical value c^s, if the change of the multipliers' summation is lower than a small positive $\varepsilon > 0$ compared to last iteration, even the solution for Δ_t should change according to (28), we kept the solution unchanged in this iteration.

2.4 Dual problem

Given the solutions of the two types of subproblems as above, the multiplier λ is to be solved for the dual problem $q(\lambda)$ under the solution of the primal problem $d_{n,t}^{(k)}, \Delta_t^{(k)}$. Because the expectation of decision variables $\mathrm{E}d_{n,t}$ and $\mathrm{E}\Delta_t$ are obtained from the two subproblems, the dual problem has a deterministic objective function on the deterministic λ, which is written as:

$$q(\lambda) = \sum_n (\mathrm{E}d_{n,t} - \mathrm{E}\Delta_t)\lambda_{n,t} + c^s \sum_t \mathrm{E}\Delta_t + \sum_n c_n^p \sum_t \mathrm{E}d_{n,t}, \quad (16)$$

where $\mathrm{E}d_{n,t} - \mathrm{E}\Delta_t$ is the sub-gradient of $q(\lambda)$, and the multiplier λ can be solved by the sub-gradient method iteratively:

$$\lambda_{n,t}^{k+1} = \lambda_{n,t}^k + s^k(\mathrm{E}d_{n,t}^k - \mathrm{E}\Delta_t^k). \quad (17)$$

The step-size s^k is a positive number satisfying the following inequality (18)[18]. If λ^k is not optimal, and all the step sizes satisfy the following inequality[18]:

$$0 < s^k < \frac{2(q^* - q(\lambda^k))}{\|g^k\|^2}, \quad (18)$$

then for every dual problem, the optimal solution λ^* verifies

$$\|\lambda^{k+1} - \lambda^*\|^2 < \|\lambda^k - \lambda^*\|^2. \quad (19)$$

In inequality (18), q^* is the optimal solution for the dual problem, conventionally it is estimated by the feasible solutions and dual solutions. For this problem, we provide a lower bound of the original problem as the estimation of q^*.

The usage of this LLP lasts for the whole time horizon except for the maintenance duration caused by failures. For an ideal case that all part replacements and maintenance visits are completely synchronized, the solution gives a lower bound of the original problem. Denote the number of replacements of part n as v_n, its initial life as $x_{n,0}$, and the number of maintenance visits as V. Then the following relationships hold:

$$(v_n - 1)S_n + (V - v_n) + x_{n,0} + 1 < T, \quad (20)$$

$$v_n S_n + (V - v_n) + x_{n,0} + 1 \geq T, \quad (21)$$

$$v_n \leq V, \quad (22)$$

It arrives that

$$V \geq \max_n \frac{T - x_{n,0} - 1}{S_n}, \quad (23)$$

$$v_n \geq \frac{T - V - x_{n,0} - 1}{S_n + 1}. \quad (24)$$

Then we have the following theorem.

Theorem 1. The lower bound of the maintenance cost is given by:

$$\underline{J} = c^s \max\left\{V, \sum_{t=0}^{T-1} r_t\right\} + \sum_n c_n^p v_n, \quad (25)$$

where

$$V = \left\lceil \max_n \frac{T - x_{n,0} - 1,}{S_n} \right\rceil, \quad (26)$$

$$v_n = \left\lceil \frac{T - V - x_{n,0} - 1}{S_n + 1,} \right\rceil, \quad (27)$$

where $\lceil\cdot\rceil$ represents rounding up to integer.

This lower bound is absolute and does not rely on the sample space, which means it is the lower bound for all sample paths rather than for expectation only. In the testing, this lower bound is shown to be effective in the solution process.

3 Numerical testing

In this section, numerical testing is performed and OLR is compared with the other two methods, OT (Optimal threshold) and OSA (One-stage analysis) by simulation with common random numbers. The average costs achieved through the three methods in the simulation runs are regarded as the expected cost. The implementation of these methods is as below:

1) OLR. At every maintenance visit, given the current part state, the decisions for the current stage till the end are optimized in the sense that the expected total cost is the lowest within the allowed iterations. Though decisions for the later stages in an expected sense may not be feasible for those stages, the decision for the current stage is feasible to implement because the part states are given. As for later decisions, upon obtaining multipliers in the previous stages, updates will be made in the consequent visits to ensure performance as well as the feasibility when part states are available.

2) OT. Take all values through the region $[1, \min_n S_n]$ as a threshold, the one under which the average cost is the lowest for all simulated sample paths is regarded as the optimal threshold th. This average cost under th is regarded as the performance index for OT, it is actually the performance limit of OT, since in practice the sample paths cannot be all simulated in advance. Consequently, the replacement decision at maintenance is given as

$$d_{n,t}^{th}(x_{n,t}) = \begin{cases} 0, & x_{n,t} > th, \\ 1, & x_{n,t} \leq th. \end{cases} \tag{28}$$

3) OSA. At every maintenance visit, the daily average cost is estimated by simulation till the next maintenance visit upon current part states. The optimal decisions are thus made in the sense of the minimal expected cost in the simulation[2]. Till the end of the contract duration, the realized cost for a sample path is counted on. The average of the realized costs are regarded as the performance index of OSA.

In OLR, the solution is not structured as in OT, and the total cost is directly optimized rather than estimated by steady state average cost as in OSA. Thus an improved performance is expected for problems without a simple form optimal policy.

In general, the more parts can be synchronously replaced, the less the total cost is. The diversity of the parts, including part lives, prices, and initial states, is the major factor that influences the replacement synchronization. Results in [16] show that OLR outperforms OT and OSA for identical parts that have the same full lives and prices but different initial states. For nonidentical parts, that have different part full lives, prices and initial states, the complexity of the policy is much higher and the method in [16] does not converge to good enough solutions. The improvement of OLR in this paper over [16] is achieved by the following changes: 1) The alternative state transition function and the corresponding solving method for LLP subproblems; 2) the solution for maintenance visit problem with Bang-bang control; and 3) the lower bound as the estimation to optimal dual to improve the step sizing in dual solution. Here we test the OLR performance on non-identical parts. The improved performance of OLR is to be examined under a nonidentical part case, which is the most usual case.

In the following, the joint replacement problem is tested on medium and practical scale testing problems. Under each parameter setting, common random numbers are used to generate sample paths to compare the average costs achieved by the three methods.

3.1 Medium scale testing

Since the OLR method replaces the coupling constraints with their expectations, an intuitive inference is that the performance is affected heavily due to randomness. After OLR's decompositions, all part subproblems are converted into a deterministic problem and the randomness of the problem is only conveyed by the multipliers to some extent. The performance of all the three methods is tested under different failure rates. Parameters of the medium scale problem are listed in Table 2.

Taking into account that the equipment hardly has failure rates that are larger than 0.1, the failure rates in the testing problem sets, problem set PS1 $(1) \sim (5)$, are in turn 0.1, 0.05, 0.02, 0.01, 0.005, with the other parameters kept the same. The mean and standard deviation of the costs under the three methods are compared in Table 3.

It can be seen from Fig. 1 that the mean of the total costs under OLR outperforms the other two methods. The lower the failure rates, the more obvious OLR's advantage is with consideration of variance. This advantage can be explained as follow. When the failure rate is low, there are only a few maintenance visits due to random failures. In each maintenance visit, the difference in replacement decision can bring significant performance differences. When the failure rate is high, there are frequent maintenance visits due to random failures. Since the ratio of visit cost due to failures increases, which is regardless of replacement decision, the room to reduce the total costs by improving the replacement decisions is limited. That can explain the reason for which the OLR has advantages but not as obvious as in low failure rates cases. As for the jet engine maintenance problem, the equipment reliability is critical and the failure rate is very low, therefore OLR is more suitable compared to the other two methods.

When the failure rates are 0.01 and 0.005, OLR has larger variance than the other methods. The reason is that on some sample paths, OLR achieves lower costs and enlarges the variance. Figs. 2 and 3 illustrate the results of the three methods on ten sample paths under failure rate 0.01, 0.005. It can be seen that OLR outperforms other methods on every sample paths. It is worth noting that the variance of the results mainly comes from the difference of sample paths, that is, from the random failure sequence.

3.2 Practical scale testing

Usually the maintenance contract duration is three to five years, but the contract price is subject to change on an annual basis. A one-year maintenance contract problem is tested as PS2 with 30 nonidentical parts. Table 4 lists out its parameters. By limiting the maximal number of iterations as 50, the results of OLR/OT/OSA are compared in Table 5. where the lower bound is the absolute lower bound $\underline{J}$, that is, of all sample paths rather than of the mean. The gap is between the mean cost and $\underline{J}$.

Notice that a minimal cost 214 is obtained by OLR under one sample path, the gap to $\underline{J}$ is 8.6 %. On the other sample paths, due to the diversified failure sequences and more visits of random failures, the costs have larger gap above the ideal lower bound $\underline{J}$.

Table 2 Parameters of PS 1

PS1	N	T	$\boldsymbol{x}^{\mathrm{T}}$	c^s	$\boldsymbol{c}^p$	$\boldsymbol{S}$	$r_t(1)$	$r_t(2)$	$r_t(3)$	$r_t(4)$	$r_t(5)$
	5	60	0	4	[1,1,1,1,1]	[18,25,31,16,27]	0.1	0.05	0.02	0.01	0.005

Nonidentical parts do not exhibit the same synchronization as identical part in [16], though the comparison of the three methods can be illustrated by the realized sample paths in Fig. 4. At the two maintenance visits of random failures on $t = 30, 76$, OLR and OT take opportunities of random failures to replace part, which leads to fewer number of maintenance visits than OSA. OLR takes into consideration different stages in the finite horizon so it saves more replacement cost than OT. Since OT goes over all possible steady thresholds on all sample paths, its performance cannot be further improved. From the sample path of OLR, at the two visits on $t = 30, 76$, the replacement follows different thresholds, that is to say, OLR provides a dynamic-threshold type of policy.

Table 3 Performance of OLR/OT/OSA under different failure rates for PS1

		OLR	OT	OSA
PS1(1)	mean	37.6	38.4	38.4
	STD	6.54	7.06	5.36
PS1(2)	mean	31.6	31.9	32.2
	STD	5.40	5.40	5.77
PS1(3)	mean	26.4	27.5	28.1
	STD	2.68	3.21	3.11
PS1(4)	mean	23.8	25.6	25.6
	STD	1.69	1.35	1.35
PS1(5)	mean	23.4	25.2	25.2
	STD	1.26	0.63	0.63

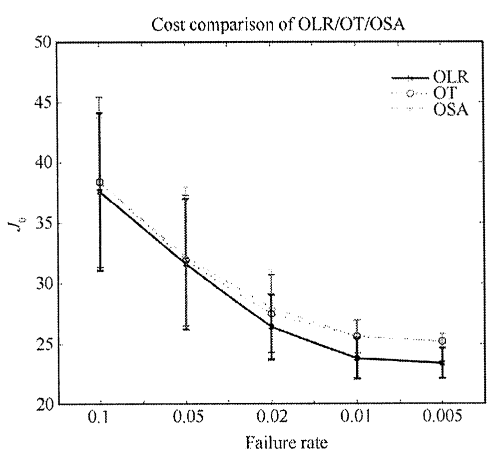


Fig. 1 Performance of OLR/OT/OSA with respect to failure rates

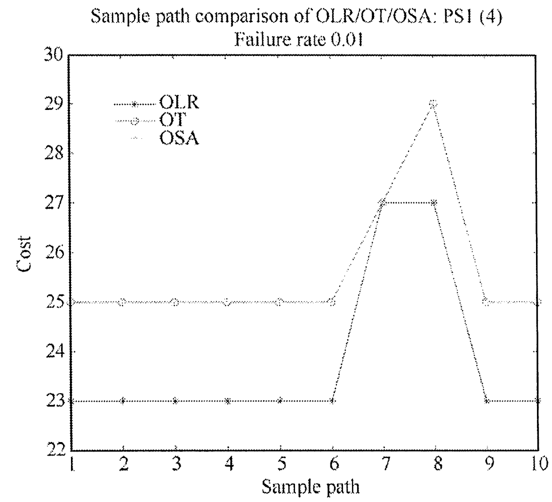


Fig. 2 Performance of OLR/OT/OSA on different samples with $r_t = 0.01$

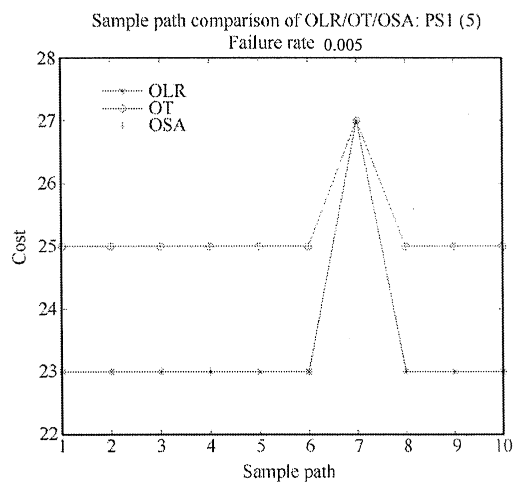


Fig. 3 Performance of OLR/OT/OSA on different samples with $r_t = 0.005$

In practice, when the replacement decision are optimized at maintenance visits, OLR requires failure rates rather than an exact failure sequence to made decisions for current stage. OT and OSA, instead, require the epochs of future failures on the sample paths to evaluate the decision for the current stage. Sampling on the future sample paths is thus required by OT and OSA, which leads to sampling errors. Though OT and OSA may obtain good results on the sample paths, there is a risk on the performance for

Table 4 Parameters of PS 2

PS2	N	r_t	T	$\boldsymbol{x}^{\mathrm{T}}$	c^s	$\boldsymbol{c^p}$	$\boldsymbol{S}$
	30	0.015	365	0	10	[1, 3, 1, 2, 3, 3, 1, 1, 1, 3, 1, 2, 1, 1, 1,2, 1, 1, 2, 1, 3, 2,]	[120, 253, 149, 132, 212, 278, 125, 249, 156, 244, 54, 212, 216, 247, 120, 128, 2, 2, 2, 1, 2, 1, 1, 3, 216, 96, 134, 277, 211, 110, 191, 190, 129, 239, 261, 196, 169, 131]

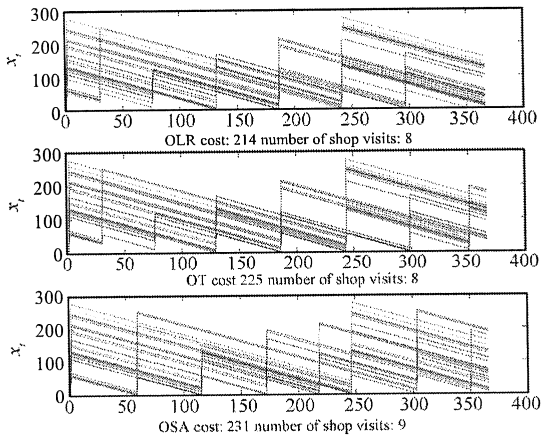


Fig. 4 Sample path comparison of OLR/OT/OSA

Table 5 Performance of OLR/OT/OSA for PS2

Method	Lower bound $\underline{J}$	Total cost		Gap
		Mean	STD	(%)
OLR	197.0	233.2	12.91	18.38
OT	–	237.2	12.35	20.41
OSA	–	240.2	14.58	21.93

particular realization. OLR provides a method for stochastic optimization problem that do not rely on sample paths, which is independent of sampling errors. This is considered as one major contribution of this paper.

4 Conclusion

This paper presents opportunistic Lagrange relaxation (OLR), a decomposition and coordination method for joint replacement problems, which is a kind of multi-stage combinatorial decision problems with uncertainties. OLR utilizes the linear characteristics of the stochastic coupling constraints to decouple the problem, thereby to avoid the simulation evaluation of current decisions on the expected cost-to-go. Since OLR does not rely on the sample space, the number of multipliers introduced to relax the coupling constraints is linear in the number of constraints and will not increase with the number of sample paths. This feature helps the large-scale problems obtain near-optimal solutions through this dual-based method. The lower bound estimation in this paper provides an estimation of the dual optimal. Controlling the iterative solution process with Bang-bang reduces the oscillation in solving the linear subproblem, so as to increase the efficiency of the dual problem. Compared to the existing algorithms OT and OSA, OLR does not depend on the sample paths, which avoids the sampling errors in decision evaluation. In the meanwhile, the multipliers introduced provide the opportunity costs for part replacement on different stages, which helps discover additional structural information. This information keeps updating the sequential decision making and increases OLR's efficiency and performance.

It is worth noticing that OLR has not utilized the structural information[3] of the optimal policy as OT and OSA have. Since OLR has the flexibility to utilize such information, it is one of the follow-up work of this paper to incorporate structural information to achieve better performance, higher algorithm efficiency and improved lower bounds.

Besides the engine maintenance problem studied in this paper, there are a lot of potential applications OLR is suitable for, such as equipment maintenance, batch ordering, and the kind of decision scheduling problems that reduce cost by sharing resources. This kind of stochastic sequential decision problems hold the following properties:

1) The decision variables are coupled;

2) The decision and state variables are coupled only on occurrence of uncertain events;

3) The coupling of the variables can be modeled through linear coupling constraints;

4) The objective of the optimization is to synchronize the variables to the maximal extent.

In addition, if the problem includes rare random events, OLR is especially effective.

It has been known that LR based methods are very effective to solve resource allocation problems, by assigning resources with multipliers that are interpreted as penalty price mechanism. The coordination problem studied in this paper, instead, is a counterpart of resource allocation. The higher the degree of synchronization is, the more costs can be reduced. Results show that the penalty price mechanism can also encourage resource sharing. It implies an interesting research direction.

References

[1] Mayer D. Life-limited parts prove a problem for FMS customers [Online], available: http://www.wpafb.af.mil/news/story.asp?id=123077131, March 31, 2012

[2] Sun T, Zhao Q C, Luh P B, Tomastik R N. Optimization of joint replacement policies for multipart systems by a rollout framework. *IEEE Transactions on Automation Science and Engineering*, 2008, **5**(4): 609−619

[3] Xia L, Zhao Q C, Jia Q S. A structure property of optimal policies for maintenance problems with safety-critical components. *IEEE Transactions on Automation Science and Engineering*, 2008, **5**(3): 519−531

[4] Jia Xi-Sheng. *The Decision Models for Reliability Centered Maintenance*. Beijing: National Defense Industry Press, 2007 (in Chinese)

[5] Sheng Tian-Wen, Chen Xiao-Hui, Yi Shu-Ping. Preventive maintenance policy for life-type equipment. *Computer In-*

tegrated Manufacturing Systems, 2009, **15**(3): 598−603 (in Chinese)

[6] Nowakowski T, Werbińka S. On problems of multicomponent system maintenance modelling. *International Journal of Automation and Computing*, 2009, **6**(4): 364−378

[7] Rao A N, Bhadury B. Opportunistic maintenance of multi-equipment system: a case study. *Quality and Reliability Engineering International*, 2001, **16**(6): 487−500

[8] Cheng Zhi-Jun, Guo Bo. Availability analysis of system under the opportunistic maintenance policy. *Mathematics in Practice and Theory*, 2006, **36**(10): 137−140 (in Chinese)

[9] Cui L R, Li H J. Opportunistic maintenance for multi-component shock models. *Mathematical Methods of Operations Research*, 2006, **63**(3): 493−511

[10] Zhang Z Q, Wu S, Li B F. A condition-based and opportunistic maintenance model for a two-unit deteriorating system. In: Proceedings of the 2011 International Conference on Quality, Reliability, Risk, Maintenance, and Safety Engineering. Xi′an, China: IEEE, 2011. 590−595

[11] Jain S. Opportunistic Maintenance Policy of a Multi-unit System under Transient State [Ph. D. dissertation], University of South Florida, USA, 2005

[12] Cheng Zhi-Jun, Guo Bo. The optimal analysis of opportunistic maintenance model of multi-components system. *Industrial Engineering Journal*, 2007, **10**(5): 66−69 (in Chinese)

[13] Rockafellar R T, Wets R J B. Scenarios and policy aggregation in optimization under uncertainty. *Mathematics of Operations Research*, 1991, **16**(1): 119−147

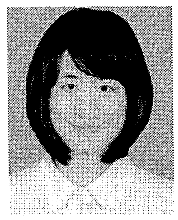
[14] Rockafellar R T, Wets R J B. A dual strategy for the implementation of the aggregation principle in decision making under uncertainty. *Applied Stochastic Models and Data Analysis*, 1992, **8**(3): 245−255

[15] Nowak M P, Römisch W. Stochastic Lagrangian relaxation applied to power scheduling in a hydro-thermal system under uncertainty. *Annals of Operations Research*, 2000, **100**(1−4): 251−272

[16] Tu G Y, Luh P B, Zhao Q C, Tomastik R N. An optimization method for joint replacement decisions in maintenance. In: Proceedings of the 43rd IEEE Conference on Decision and Control. Nasau, Bahamas: IEEE, 2004. 3674−3679

[17] Rong Xiang, Zuo Hong-Fu, Zhang Hai-Jun. Control method of time on wing for civil aeroengine. *Journal of Traffic and Transportation Engineering*, 2008, **8**(3): 28−32 (in Chinese)

[18] Bertsekas D P. *Nonlinear Programming*. New York: Athena Scientific, 2003. 610−612



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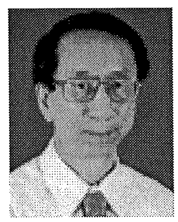

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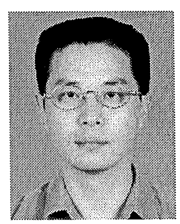

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