

Quadratic incentive coordination for non-convex optimal control problems

Part 1. Theory

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Non-convex optimal control problems are decomposed and coordinated by using quadratic incentive functions. The idea is to decompose the original problem into subproblems along the time axis so that subproblems can be solved in parallel. To coordinate the subproblems, a high-level problem is used. An equivalence is demonstrated by showing there is a one-to-one correspondence between the local minima of the high-level problem and local minima of the original problem, provided certain technical conditions are satisfied. The theoretical foundation of quadratic incentive coordination is established. The relationship of the quadratic incentives and the optimal cost-to-go functions is also established.

1. Introduction

Many methods have been developed to improve the computational efficiency of determining the solutions of long time horizon optimal control problems. The best methods, such as differential dynamic programming (Yakowitz and Rutherford 1984), both relieve the curse of dimensionality and offer quadratic convergence rates. One is not likely to improve on these methods except by relying on algorithms which require the use of higher-order derivatives.

Parallel processing, however, is one area in which one might expect to significantly improve the computational efficiency of these algorithms. The use of parallel processing is becoming increasingly significant as parallel processors become more available and more economical to the average user. Unfortunately, the decomposition of optimal control problems into disjoint subproblems is difficult and until recently most methods utilized the special structure of a given system. A complimentary approach to existing methods based upon time decomposition has been proposed by Chang and Luh (1985). Since their work, two coordination methods—incentive and target coordination—have been studied for convex problems (Chang *et al.* 1987, Chang *et al.* 1988). For non-convex problems, a parallel computation algorithm based upon the time decomposition and quadratic incentive coordination has been proposed by Chang (1987).

Motivated by these studies, the present work investigates the use of quadratic incentive terms to decompose an optimal control problem into a series of subproblems. The equivalence of the decomposed problem and the original problem can be proved in a similar sense to that given by Chang *et al.* (1987). Roughly speaking, there is a one-to-one correspondence between the local minima of the decomposed problem and that of the original one under certain technical conditions.

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Remark 1

From the Appendix one can see that each single stage of problem (P), may consist of T substages of some original problem. It is reasonable to assume that for large T , the original T -stage subproblem is T -stage controllable (Chang *et al.* 1988) or equivalently the dynamic equation (2.1 b) in problem (P) is one-stage controllable. This is because as one increases the size of T , the dimension of u_j increases while the dimension of the state x_{j+1} , remains constant.

Remark 2

Notice that Assumption 2 will be true at least locally by the implicit function theorem, if $\partial f_j / \partial u_{Dj}$ is invertible. Also note that by differentiating Assumption 2 with respect to x_{j+1} , one concludes that Assumption 2 implies that $\partial f_j(x_j, u_j) / \partial u_{Dj}$ is invertible. In addition it is important to note that equation (2.2) implies that $f_j(\cdot)$ is an open map. (It maps open sets to open sets.) Thus if U_j is an open set, $f_j(x_j, U_j)$ is an open set, where $f_j(x_j, U_j) \equiv \{f_j(x_j, u_j), u_j \in U_j\}$.

Remark 3

Assumption 3 will be used to justify the idea that in trying to minimize $J(u)$ one can restrict our search to u (and x) which lie in some compact set. For any iterative scheme which generates successive u (or x) which decreases J , the u (or x) must all lie in some compact set. Thus one can assume that for solving the problems which follow, all u of interest and thus all x , lie in some compact ball B_u or B_x . Note that any function which is compact below is also bounded below.

As given by Chang *et al.* (1987), one can decompose (P) into subproblems along the time axis and coordinate them by using incentives. For non-convex problems, one needs to use quadratic incentives instead of the linear ones used for convex problems (Chang *et al.* 1987). The hierarchical decomposition can then be described as follows.

Incentive coordination: (P - H)

$$\min_p J^I(p), \quad J^I(p) = \sum_{j=0}^{M-1} g_j(x_j^*, u_j^*) \tag{2.3 a}$$

where

$$p \equiv (p_0, p_1, \dots, p_{M-1}) \in \mathbb{R}^{Mn} \tag{2.3 b}$$

and x_j^* and u_j^* are the optimal solutions for the low-level subproblems defined below.

Subproblem: (P - j)

$$\min_{u_j} J_j^I(u_j | x_j, p_j)$$

$$J_j^I(u_j | x_j, p_j) \equiv g_j(x_j, u_j) + I_j(x_{j+1}) \tag{2.4 a}$$

where

$$I_j(x_{j+1}) \equiv \frac{1}{2} Q x_{j+1}^T x_{j+1} + p_j x_{j+1} \tag{2.4 b}$$

and

$$x_{j+1} = f_j(x_j, u_j) \tag{2.4 c}$$

The solution to (2.4 a) will be denoted by $u_j^* = u_j^*(x_j, p_j)$. Note that Q is a given positive real parameter to be determined later and p_j is the incentive coefficient. Our goal is then to develop parallel algorithms to obtain the optimal p_j^* iteratively such that the low-level optimal solutions are also the optimal solution of the original problem. In the next section, by assuming that the subproblems are solved sequentially, the existence of such incentive terms can be demonstrated. In other words, problem (P-j) will be solved first to obtain its solution. The optimal terminal solution of (P-j) is then used as the initial condition of (P-(j+1)). Since in this case x_j would be determined by the previous incentives $p_i, 0 \leq i < j$, one concludes that for each conglomerate vector $p \equiv (p_0, p_1, \dots, p_{M-1})$, all the u_j^* are determined. Therefore this dependency will often be indicated by $u_j^*(p)$.

There is one major difference between the two-level decomposition for convex and non-convex problems. For convex problems, one can find global optimal solutions for both the high level and the low level. Thus the two-level decomposition is well defined mathematically. For non-convex problems, the best one can expect for existing deterministic algorithms is to obtain local minima. This implies that $J^T(p)$ in (2.3) is not well defined since for a given value of p the local minimum for a non-convex (P-j) is typically not unique. Additional assumptions are then needed to define the two-level problem mathematically.

Given a fixed algorithm one can determine local minimizers $u_j^*(p)$ for $J_j^T(u_j | x_j, p_j)$ as a function of u_j . These local minima will be global minima in small open balls about $u_j^*(p)$.

Definition 1: $r_j(p)$

Let $r_j(p)$ be a radius function chosen so that $J_j^T(u_j)$ has a global minimum at $u_j^*(p)$ in $B(u_j^*(p), r_j(p))$ —the open ball with centre at $u_j^*(p)$ and radius $r_j(p)$. The ball in which this global minimum occurs will be referred to as $B_j(p)$. It is convenient to assume that the minimizer $u_j^*(p)$ is also a global minimizer for the closed ball $\overline{B_j(p)}$ —this can be achieved by shrinking the radius, $r_j(p)$ slightly. In this notation the dependency on p is used to indicate the fact that the location of the local minimum and the set in which it is a global minimizer depends on the value of p . Using this notation one can write

$$u_j^*(p) = \arg \min_{u_j \in B_j(p)} (g_j(x_j, u_j) + l_j(x_{j+1})) \quad (2.4 d)$$

As noted by Chang *et al.* (1987), the target coordination given by Chang *et al.* (1988) and Chang *et al.* (1986) can be used as an independent coordination scheme. It can also be considered as the foundation of developing the theory for the incentive coordination scheme. Let us define the target coordination scheme as follows.

Target coordination: (PT - H)

$$\min_x J^T(x) \\ J^T(x) = \sum_{j=0}^{M-1} g_j(x_j^*, u_j^*) \equiv \sum_{j=0}^{M-1} g_j^*(x_j, x_{j+1}) \quad (2.5)$$

where x_j^* and u_j^* are the optimal solutions for the low-level subproblems and

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$g_j^*(x_j, x_{j+1})$, denote the optimal cost of (PT - j) defined below.

Subproblem: (PT - j)

$$\min_{u_j} J_j^T(u_j | x_j, x_{j+1}) \tag{2.6 a}$$

where

$$J_j^T(u_j | x_j, x_{j+1}) = g_j(x_j, u_j) \tag{2.6 b}$$

$$= g_j(x_j, u_{Dj}(x_j, x_{j+1}, u_{Ij}), u_{Tj}) \equiv \hat{g}_j(x_j, x_{j+1}, u_{Ij}) \tag{2.6 c}$$

$$x_{j+1} = f_j(x_j, u_j) \text{ and both } x_j \text{ and } x_{j+1} \text{ are given} \tag{2.6 d}$$

In practice, different but fixed algorithms are used for the high- and low-level subproblems. Since the state of art deterministic algorithms for non-convex problems can only find local minima, it will be assumed for theoretical purposes that all the optimization operations are in terms of local minima. In other words, when one says $u_j^*(x_j, p_j)$ is the optimal solution to (P - j), one means that

$$u_j^*(x_j, p_j) \equiv \arg \min_{u_j} J_j^I(u_j | x_j, p_j) \tag{2.7}$$

where 'arg' indicates that it is the solution of the problem and 'lmin' means that it is the local minimum generated by the algorithm solving (P - j).

Since there are quite a few subproblems involved in the present work, the notation used is rather complicated. Our notational conventions are described below. The superscript '*' on a given variable always represents that variable or function names' restriction to the optimal solutions of a given subproblem. The superscripts 'P', 'I' and 'T', are used to represent the basic problem, the incentive coordination and the target coordination, respectively, and their individual subproblems. Although their functional dependencies may vary, a given quantity will be denoted by the same variable, and the functional dependencies will be indicated by the variable list which follows the function name. From the variable list which follows a function name it will often be possible to determine what subproblem type the variables come from. For function composition, unfortunately, one must rely on the context or resort to the use of vertical bars to indicate function evaluation. Also for the variables p_j , x_j and u_j and their various functional dependencies we will form conglomerate vectors by concatenating the vectors from each stage together. This will be indicated by dropping the subscript from the given variable. For quick reference, some of the notation used is listed in the Table.

(a)	(b)	(c)	(d)	(e)	(f)
P	u_j^*, x_j^*	u^*, x^*	u_j^*, x_j^*	$J_j^{P^*}(x_{j+1})$	$J(u)$
I	$u_j^*(x_j, p_j), x_{j+1}^*(x_j, p_j)$	$u^*(p), x^*(p)$	$u_j^*(p), x_{j+1}^*(p)$	$J_j^{I^*}(x_j, p_j)$	$J^I(p)$
T	$u_j^*(x_j, x_{j+1})$	$u^*(x)$	$u_j^*(x)$	$J_j^{T^*}(x_j, x_{j+1})$	$J^T(x)$

Summary of notation.

(a) A row of variables associated with the problem type; (b) optimal control and state trajectories associated with subproblem j; (c) conglomerate control and state trajectories for (b); (d) coordinate vector functions for the conglomerate vector functions in (c); (e) optimal solution for each subproblem, as a function of given parameters; (f) high-level cost function associated with each high-level problem.

Note that $J_j^{p^*}(x_{j+1})$ represents the cost-to-go function of (P) given by

$$J_j^{p^*}(x_j) = \min_{u_k} J_j^p(x_j, u_{(j, M-1)}) \equiv \sum_{k=j}^{M-1} g_j(x_j, u_j) \tag{2.8 a}$$

where

$$x_{k+1} = f_k(x_k, u_k), \quad j \leq k < M, \quad x_j \text{ is given} \tag{2.8 b}$$

and by definition

$$u_{(j, M-1)} \equiv (u_j, u_{j+1}, \dots, u_{M-1}) \tag{2.8 c}$$

Also note that $u_j^*(x_j, x_{j+1})$ can be decomposed into two components from Assumption 2, so that one can write

$$u_j^*(x_j, x_{j+1}) = (u_{Dj}(x_j, x_{j+1}), u_{Ij}^*(x_j, x_{j+1})) \tag{2.9}$$

3. Equivalence of quadratic incentive coordination

The objective of this section is to prove that the local minima of the subproblems (P-j), dependent upon a given incentive p^* —a local minimum of the high-level problem (P-H)—together form a local minimum of the basic problem (P). Conversely, for a given local minimum u^* of the basic problem (P), which can be generated by some incentive p^* , p^* is a local minimum of (P-H). Thus if u^* is a local minimum of $J(u)$ and there exists a p^* such that $u^* = u^*(p^*)$, then p^* is a local minimum of $J^I(p)$. In other words, the decomposed problem with quadratic incentives is equivalent to the original problem in a similar sense to that stated by Chang *et al.* (1987).

First, it is demonstrated that quadratic incentives are sufficient to control the location of the minimizer of a function of a state variable to any desired location (Lemma 1). In Lemma 2, it is shown that for sufficiently small neighbourhoods in P , solving for $u^*(p)$ amounts to finding global minima of the $J_j^I(\cdot)$ in a small fixed neighbourhood contained in U . With this tool, the equivalence between a local minimum for the high-level problem and a local minimum for the basic problem, will be demonstrated in Theorem 1. With additional assumptions about the differentiability of optimal solutions, it can be shown that the conglomerate state and incentive vectors are in a one-to-one correspondence via differentiable functions (Lemma 3). This also leads to a sufficient condition for the equivalence of the high-level and the original problems in Theorem 2.

Lemma 1

If $f(x), \mathbb{R}^n \rightarrow \mathbb{R}$, is C^2 in a closed ball B , then for all y in B there exists a quadratic incentive

$$I(x) = \frac{1}{2} Qx^T x + px \tag{3.1}$$

such that

$$\arg \min_{x \in B} (f(x) + I(x)) = y \tag{3.2}$$

and the solution y is unique.

Proof

Denote differentiation say with respect to x by the subscript x , and the identity matrix by I . Now the matrix sup norm $|M|$, is a continuous function of $n \times n$ matrices

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M , to the positive real line and by definition is the largest of the norms of the eigenvalues of M . Since $f_{xx}(x)$ is continuous, so is the function $|f_{xx}(x)|$, $\mathbb{R}^n \rightarrow \mathbb{R}$. Thus since B is a compact set, so is $|f_{xx}(B)|$ and a Q can be chosen, so that $Q > |f_{xx}(x)|$ for all x in B . Now choose

$$p = -Qy - f_x(y) \quad (3.3)$$

and define

$$h(x) \equiv f(x) + I(x) \quad (3.4)$$

With the above definitions note that

$$h_x(y) = f_x(y) + p + Qy = 0 \quad (3.5)$$

and that

$$h_{xx}(z) = f_{xx}(z) + Q \cdot 1 > 0 \quad (3.6)$$

for all z in B . Thus by the Taylor theorem with the Lagrange form of the remainder, there exists a z on the line segment between x and y such that

$$h(x) = h(y) + h_x(y)(x - y) + \frac{1}{2}(x - y)' h_{xx}(z)(x - y) \quad (3.7)$$

This is justified since if x and y are in B then so is the line segment from x to y . By (3.5) and (3.6), one can conclude that

$$h(x) - h(y) = (x - y)' h_{xx}(z)(x - y) > 0 \quad (3.8)$$

when x is not y in B . This shows that

$$y = \arg \min_{x \in B} h(x) \quad (3.9)$$

as desired and this minimum is unique. \square

Note that in the proof above the choice of Q is independent of y . It is necessary to assume the following notion of continuity for the local minima generated by the algorithm solving the (P - j).

Assumption 4: Continuous algorithm

The algorithm is called continuous if and only if $r_j(p)$ and $u_j^*(p)$ are continuous functions of p . The present work assumes the algorithm is continuous.

Definition 2: F(u) conglomerate system dynamics

Let $F(u)$ be the function which maps the controls u to the conglomerate end states x via the system dynamics. Thus $F(u)$ is defined iteratively as follows:

$$F(u) \equiv (x_1, x_2, \dots, x_M) \quad (3.10)$$

where $x_{j+1} = f_j(x_j, u_j)$ and x_0 is given. Since the $f_j(\cdot)$ are smooth functions and $F(u)$ is a composition of these functions, $F(u)$ is smooth. By Remark 2 and (2.2) it is a simple matter to prove that $F(u)$ is an open map.

Definition 3: $x^(p)$ incentive-to-state map*

$$x^*(p) \equiv F(u^*(p)) \quad (3.11)$$

With Assumption 4, at once it is clear that $x^*(p)$ is continuous. In what follows, if S is an arbitrary set and f an arbitrary function, define the set $f(S) \equiv \{f(s) : s \in S\}$.

Lemma 2

If the algorithm is continuous as in Assumption 4, then for any p in P , there exists an open neighbourhood Π_p such that p is in Π_p and open neighbourhoods $U_j^*(p)$ of $u_j^*(p)$ such that if \bar{p} is in Π_p then

$$u_j^*(\bar{p}) = \arg \min_{u_j \in U_j(p)} J_j^f(x_j^*(\bar{p}), u_j) \tag{3.12}$$

Also letting $U(p) \equiv (U_0(p), U_1(p), \dots, U_{M-1}(p))$, one concludes $u^*(\Pi_p)$ is a subset of $U(p)$; and if T is an open set containing p , then Π_p can be chosen so that Π_p is a subset of T .

Proof

Note that p is fixed throughout. In order to find the local minimum $u_j^*(\bar{p})$ in (3.12) for any \bar{p} in the neighbourhood of p , it is only necessary to search for it in a fixed open set $U_j(p)$ —for any \bar{p} . In other words, the significance of this theorem lies in the fact that, at least locally, a continuous algorithm does nothing more than find a global minimum in a certain fixed set $U_j(p)$.

Choose a ball of radius s about p so that $B(p, s)$ is a subset of T . Let I be the closure of $B(p, s/2)$ so that I is a compact set with non-empty interior. Call the interior $I^0 = B(p, s/2)$. Let $r_j^* = \min_{p \in I} r_j(p)$. Now by Assumption 4 $r_j(p)$ is continuous and I is compact. Thus $r_j(I)$ is a compact subset of the positive real line. The minimum is thus achieved in I at some point \bar{p} . Since it is assumed our algorithm always finds a local minimum for J_j^f , then $r_j(\bar{p}) = r_j^* > 0$ follows. Let

$$U_j(p) = B\left(u_j^*(p), \frac{r_j^*}{2}\right)$$

Let

$$U(p) \equiv (U_0(p), U_1(p), \dots, U_{M-1}(p))$$

be the cartesian product of the $U_i(p)$. Here $U(p)$ is an open set. Define

$$B_j(p) \equiv B(u_j^*(p), r_j(p))$$

and define

$$\Pi_p \equiv u^{*-1}(U(p)) \cap I^0$$

Notice p is in Π_p , thus Π_p is non-empty. Since $u^*(\bar{p})$ is continuous, $u^{*-1}(U(p))$ is open, thus Π_p is open. Let $K \equiv x^*(\Pi_p)$ and $M \equiv F^{-1}(K)$. This yields the following picture.

It can be shown that

$$U_j(p) \text{ is a subset of } \bigcap_{\bar{p} \in \Pi_p} B_j(\bar{p}) \tag{3.13}$$

This is proved as follows. Suppose $\bar{u}_j \in U_j(p)$, then by definition

$$|u_j^*(p) - \bar{u}_j| < \frac{r_j^*}{2} \tag{3.14}$$

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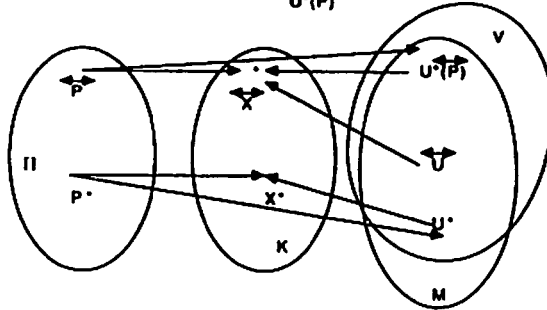
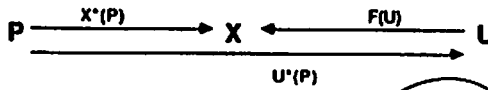
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However if $\bar{p} \in \Pi_p = u^{*-1}(U(p)) \cap I^0$ then $u^*(\bar{p})$ is in $U(p)$, which implies that $u_j^*(\bar{p})$ is in $U_j(p)$. Thus

$$|u_j^*(p) - u_j^*(\bar{p})| < \frac{r_j^*}{2} \tag{3.15}$$

So it follows that,

$$|\bar{u}_j - u_j^*(\bar{p})| < |u_j^*(p) - \bar{u}_j| + |u_j^*(p) - u_j^*(\bar{p})| < \frac{r_j^*}{2} + \frac{r_j^*}{2} = r_j^* \tag{3.16}$$

However, \bar{p} is in I^0 which is a subset of I . Hence by the definition of r_j^* , $r_j^* \leq r_j(\bar{p})$ which yields, $|\bar{u}_j - u_j^*(\bar{p})| \leq r_j^* \leq r_j(\bar{p})$. Thus \bar{u}_j is in $B_j(\bar{p})$ by definition of $B_j(\bar{p})$. Since \bar{p} was chosen arbitrarily in Π_p and \bar{u}_j was an arbitrary element of $U_j(p)$, (3.13) is established.

Now Π_p is a subset of $u^{*-1}(U(p))$ so $u^*(\Pi_p)$ is a subset of U . Also $\Pi_p \subseteq I^0 \subseteq I \subseteq B(p, s) \subseteq T$ as desired. By the definition of $r_j(\bar{p})$, J_j^i has a global minimum at $u_j^*(\bar{p})$ in $B_j(\bar{p})$. Thus from (3.13) J_j^i has a global minimum at $u_j^*(\bar{p})$ in $U_j(p)$. This yields

$$u_j^*(\bar{p}) = \arg \min_{u_j \in U_j(p)} J_j^i(x_j^*(p), u_j) \tag{3.17}$$

when $\bar{p} \in \Pi_p$. Thus the theorem is proved. □

The next theorem will show that it is appropriate to define the high-level cost as $J^I(p) \equiv J(u^*(p))$. Now it can be demonstrated that a local minimum for $J^I(p)$ in p corresponds to a local minimum for $J(u)$ in u . It is possible to show this by only assuming that $x^*(p)$ is an open map and that the given algorithm is continuous.

Theorem 1

Assuming a continuous algorithm, suppose $x^*(p)$ is an open map and $u^* = u^*(p^*)$. Then $J(u^*(p)) = J^I(p)$ has a local minimum at p^* if and only if $J(u)$ has a local minimum at u^* .

Proof

First assume that the algorithm is continuous, so that the conditions of Lemma 2 are satisfied and that $x^*(p)$ is an open map. Let T be an open neighbourhood about p^*

in P in which $J^1(p)$ has a global minimum at p^* . Let T be associated with the T in Lemma 2. Then it is claimed that $J(u)$ has a global minimum at $u^* = u^*(p^*)$ in

$$U(p^*) \cap M \equiv U(p^*) \cap F^{-1}(x^*(\Pi_{p^*}))$$

where Π_{p^*} , $U(p^*)$, and M are generated as in Lemma 2. Since Π_{p^*} is open and $x^*(\cdot)$ is an open map $x^*(\Pi_{p^*})$ is an open set and the fact that $F(\cdot)$ is continuous yields that M is an open set. For any given $\bar{u} \in U(p^*) \cap M$, it is desirable to show that $J(\bar{u}) \geq J(u^*(p^*))$. Since $\bar{u} \in M$ there exists a \bar{p} in Π_{p^*} such that $F(\bar{u}) = \bar{x} = x^*(\bar{p})$. By Lemma 2 $u^*(\Pi_{p^*})$ is a subset of $U(p^*)$. This then yields $u^*(\bar{p}) \in U(p^*)$. However, by the definition of $x^*(p)$, $F(u^*(\bar{p})) = x^*(\bar{p}) = \bar{x}$ which implies $u^*(\bar{p}) \in M$. Thus the situation in the Figure follows. Since $U(p^*) \cap M$ is a subset of $U(p^*)$ one concludes that \bar{u}_j and $u_j^*(\bar{p})$ are elements of $U_j(p^*)$. By Lemma 2

$$u_j^*(\bar{p}) = \arg \min_{u_j \in U_j(p^*)} J_j^1(\bar{x}_j, u_j) \tag{3.18}$$

This implies that

$$J_j^1(\bar{x}_j, \bar{u}_j) \geq J_j^1(\bar{x}_j, u_j^*(\bar{p})) \tag{3.19}$$

which yields

$$g_j(\bar{x}_j, \bar{u}_j) + I_j(\bar{x}_{j+1}) \geq g_j(\bar{x}_j, u_j^*(\bar{p})) + I_j(\bar{x}_{j+1}) \tag{3.20}$$

The fact that

$$\bar{x} = F(u^*(\bar{p})) = F(\bar{u}) \tag{3.21}$$

is being used here. Thus

$$g_j(\bar{x}_j, \bar{u}_j) \geq g_j(\bar{x}_j, u_j^*(\bar{p})) \tag{3.22}$$

Summing over j yields

$$J(\bar{u}) \geq J(u^*(\bar{p})) = J^1(\bar{p}) \tag{3.23 a}$$

However $J^1(p)$ has a global minimum in T at p^* . By Lemma 2, Π_{p^*} is a subset of T . Therefore J^1 has a global minimum in Π_{p^*} at p^* . Since $\bar{p} \in \Pi_{p^*}$ one concludes that

$$J^1(\bar{p}) \geq J^1(p^*) = J(u^*(p^*)) = J(u^*) \tag{3.23 b}$$

So from (3.23) it can be shown that

$$J(\bar{u}) \geq J(u^*) \tag{3.24}$$

Now since $F(u^*) = x^* = x^*(p^*)$, one concludes, $u^* \in F^{-1}(x^*(p^*))$. Thus $u^* \in M$. However, by the definition of $U(p)$ in Lemma 2, $u^* \in U(p^*)$ as well. Thus $U(p^*) \cap M$ is not empty and J has a global minimum there at u^* , since $J(\bar{u}) \geq J(u^*)$ for arbitrary \bar{u} in $U(p^*) \cap M$. Thus $J(u)$ has a local minimum at u^* .

Conversely if $J(u)$ has a local minimum at $u^* = u^*(p^*)$ then $J(u)$ has a global minimum at u^* in some open neighbourhood U about u^* . By Assumption 4,

$$u^*(p) \equiv (u_0^*(p), u_1^*(p), \dots, u_{M-1}^*(p))$$

is a continuous function of p . Therefore $u^{*-1}(U)$ is an open set in P containing p^* . If $\bar{p} \in u^{*-1}(U)$, then $u^*(\bar{p}) \in U$ and by definition of u^* one can write,

$$J^1(\bar{p}) = J(u^*(\bar{p})) \geq J(u^*) = J^1(p^*) \tag{3.25}$$

Thus p^* is the global minimizer of $J^1(p)$ in $u^{-1}(U)$ and thus $J^1(p)$ has a local minimum at p^* . □

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The notation established in Lemma 2 will be used throughout this paper. For Lemma 3, the following concept of the local differentiability of solutions to (P-j) will be needed. This will provide a sufficient condition for the map $x^*(p)$ to be an open map.

Definition 4: C^n solutions

The basic problem has C^n solutions if and only if for any given \bar{p} and $x \in F(U(\bar{p}))$, the following function

$$g_j^*(x_j, x_{j+1}) \equiv \min_{u_j \in S_j} g_j(x_j, u_j) \tag{3.26}$$

is C^n , where S_j is defined by

$$S_j \equiv \{u_j : u_j \in U_j(\bar{p}) \text{ and } f_j(x_j, u_j) = x_{j+1}\} \tag{3.27 a}$$

Also define T_j by

$$T_j \equiv f_j(x_j, U_j(\bar{p})) \tag{3.27 b}$$

Note that T_j is open and by the definition of $U_j(\bar{p})$, the closure of $U_j(\bar{p})$ is compact and thus T_j lies in a compact set. Define this compact set as

$$\bar{T}_j \equiv f_j(x_j, \overline{U_j(\bar{p})}) \tag{3.27 c}$$

The minimization in (3.26) is equivalent to minimizing $g_j(x_j, x_{j+1}, u_{1j})$ over u_{1j} using some local minimum finding algorithm. The solution $u_{1j}^*(x_j, x_{j+1})$ will depend on x_j, x_{j+1} , and the initial control u_{1j}^0 used to start the iterative algorithm. This implies that

$$\begin{aligned} u_j^*(x_j, x_{j+1}) &\equiv (u_{Dj}^*(x_j, x_{j+1}), u_{1j}^*(x_j, x_{j+1})) \\ &\equiv (u_{Dj}(x_j, x_{j+1}, u_{1j}^*(x_j, x_{j+1})), u_{1j}^*(x_j, x_{j+1})) \end{aligned} \tag{3.28 a}$$

and

$$g_j^*(x_j, x_{j+1}) \equiv g_j(x_j, u_j^*(x_j, x_{j+1})) \tag{3.28 b}$$

Lemma 3

If the basic problem has C^n solutions, $n \geq 2$, then for any given \bar{p} , there exists a one-to-one correspondence between p and x via C^{n-1} functions $p^*(x) = p$ defined on the set $x^*(\Pi_p)$ and $x^*(p) = x$ defined on Π_p . Furthermore $x^*(\Pi_p)$ is an open set.

Proof

The one-to-one correspondence of x and p can be seen as follows. For any $p \in \Pi_p$, the solution can be written—recall that $I_j(x_{j+1})$ is dependent on p —as

$$u_j^*(p) = \arg \left\{ \min_{u_j \in U_j(\bar{p})} (g_j(x_j, u_j) + I_j(x_{j+1})) \right\} \tag{3.29}$$

$$= \min_{x_{j+1} \in T_j} \min_{u_j \in S_j} (g_j(x_j, u_j) + I_j(x_{j+1})) \tag{3.30}$$

$$= \min_{x_{j+1} \in T_j} (g_j^*(x_j, x_{j+1}) + I_j(x_{j+1})) \tag{3.31}$$

Since the local minimum is a global minimum in $U_j(\bar{p})$, (3.30) holds, where S_j is

defined in (3.27 a). Thus by (3.26), (3.31) results. Also by Definition 4 $g_j^* + I_j(x_{j+1})$ is at least C^2 . One can now apply Lemma 1 to (3.31) as a function of x_{j+1} . A single Q can be chosen to force $g_j^*(x_j, x_{j+1}) + I_j(x_{j+1})$ to a unique global minimum in $T_j(p)$, which is compact. (By Definition 1 we can use closed and compact sets instead of the open sets used here.) By Lemma 2, (3.29) is valid for all $p \in \Pi_p$. It follows from Lemma 1 that one can induce any x_{j+1} in $x^*(\Pi_p)$ as the solution for (3.31) by choosing an appropriate p in Π_p . Since the solution is unique for (3.31), the one-to-one correspondence between x and p is thus established. Actually, the relationship between them can be found explicitly as follows.

The necessary conditions for (3.31) are

$$\frac{\partial g_j^*(x_j, x_{j+1})}{\partial x_{j+1}} + p_j + Qx_{j+1} = 0 \tag{3.32}$$

The dependency for p_j can then be expressed by

$$p_j = p_j^*(x_j, x_{j+1}) \equiv \frac{\partial g_j^*(x_j, x_{j+1})}{\partial x_{j+1}} - Qx_{j+1} \tag{3.33}$$

Note $p_j^*(\cdot)$ is a C^{n-1} function of x_j and x_{j+1} . This yields

$$p = p^*(x) \equiv (p_0^*(x_0, x_1), p_1^*(x_1, x_2), \dots, p_{M-2}^*(x_{M-2}, x_{M-1})) \tag{3.34}$$

Thus $p^*(x)$ is a C^{n-1} function of x . This also shows $x^*(p)$ is one-to-one on Π_p since its inverse $p^*(x)$ has been exhibited. One can therefore write $x^*(p) = p^{*-1}(p)$.

To see that $x^*(\Pi_p)$ is an open set, it has been noted earlier that the minimum in (3.31) is unique in a compact set, provided Q is large enough. Any x which satisfies the necessary conditions (for all j), in (3.33) will be a solution to (3.31). The x which satisfy these conditions as p ranges over all possible values in Π_p is simply $p^{*-1}(\Pi_p)$. Since $n \geq 2$, $p^*(x)$ is at least C^1 and therefore continuous. This implies that $p^{*-1}(\Pi_p)$ is an open set since Π_p is open. This in turn implies that $x^*(\Pi_p)$ is open because

$$x^*(\Pi_p) = p^{*-1}(\Pi_p) \tag{3.35}$$

To show that $x^*(p)$ is a C^{n-1} function, we have from the C^{n-1} function $p^*(x)$

$$\frac{\partial p_j^*(x_j, x_{j+1})}{\partial x_{j+1}} = - \frac{\partial^2 g_j^*(x_j, x_{j+1})}{\partial x_{j+1}^2} - Q \tag{3.36}$$

our consideration can be restricted to the compact set $F(B_u)$ and $\partial^2 g_j^*(x_j, x_{j+1})/\partial x_{j+1}^2$ is continuous, one can see that the eigenvalues of $\partial^2 g_j^*(x_j, x_{j+1})/\partial x_{j+1}^2$ lie in a compact set. (This can be seen by noting that the numerical range of $\partial^2 g_j^*(x_j, x_{j+1})/\partial x_{j+1}^2$ is a continuous function of a compact set.) Thus one can choose a single large fixed Q so that

$$\frac{\partial p_j^*(x_j, x_{j+1})}{\partial x_{j+1}} = \frac{\partial^2 g_j^*(x_j, x_{j+1})}{\partial x_{j+1}^2} - Q \cdot 1 < 0 \tag{3.37}$$

for all x in $F(B_u)$ and for all j . Thus $\partial p_j^*(x_j, x_{j+1})/\partial x_{j+1}$ is invertible for all x of interest. Therefore by the implicit function theorem there exists a C^{n-1} function, $x_{j+1}^*(x_j, p_j)$ so that $p_j = p_j^*(x_j, x_{j+1}^*(x_j, p_j))$. The function will exist locally about any point (x_j, p_j) , however, on any overlapping region where the p_j and x_j are the same, they will have the same solution x_{j+1} of (3.36), since by Lemma 1, the solution to this is unique. Thus the $x_{j+1}^*(x_j, p_j)$ must be equal on the overlapping regions. One can

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recursively obtain functions $x_j^*(x_{j-1}, p_{j-1}), x_{j-1}^*(x_{j-2}, p_{j-2}) \dots$, and so on until x_{j+1} is expressed as a C^{n-1} function of p and x_0 which is fixed. This generates a C^{n-1} function $\bar{x}(p)$. However, since $p^*(\bar{x}(p)) = p^*(x^*(p))$ and p is one-to-one, $\bar{x}(p) = x^*(p)$ and thus $x^*(p)$ is C^{n-1} . \square

The previous lemma guarantees the existence of a C^{n-1} function $p^*(x)$, which has the inverse $x^*(p)$ when restricted to the sets $x^*(\Pi_{\bar{p}})$, $\bar{p} \in P$. In addition, it was demonstrated that $x^*(\Pi_{\bar{p}})$ is an open set for any $\bar{p} \in P$. An examination of Theorem 1 reveals that this is sufficient to guarantee us equivalence of local minima as seen from the following corollary.

Corollary 1

If the basic problem has at least C^2 solutions, then $x^*(p)$ is an open map.

Proof

Let H be an open set in P ; H can be written as

$$H = \bigcup_{\bar{p} \in P} \Pi_{\bar{p}} \cap H \tag{3.38}$$

This is because $\Pi_{\bar{p}}$ contains \bar{p} and therefore the union covers all P . By Lemma 3 $x^*(p)$ has a continuous (C^1) inverse $p^*(x)$ on the open set $x^*(\Pi_{\bar{p}})$. This implies that

$$x^*(\Pi_{\bar{p}} \cap H) = p^{*-1}(\Pi_{\bar{p}} \cap H) \tag{3.39}$$

is an open set, since $\Pi_{\bar{p}} \cap H$ is open. However, a function of a union of sets is the union of the function of each set, therefore by (3.38) we have

$$x^*(H) = \bigcup_{\bar{p} \in P} x^*(\Pi_{\bar{p}} \cap H) \tag{3.40}$$

Since the union of open sets is an open set, by (3.39) $x^*(H)$ is an open set. Since H was arbitrary, this implies that $x^*(p)$ is an open mapping. This condition along with the underlying assumption that $u^*(p)$ and the $r_j(p)$ are continuous is all that is needed to guarantee the conclusions of Theorem 1. \square

Theorem 2

Assuming a continuous algorithm, if the basic problem (P) has C^n solutions, $n \geq 2$, then problems (P) and (P - H) are equivalent in the sense of Theorem 1.

4. Incentive coordination and optimal cost-to-go

One would now like to interpret the solution of problem (P - H) in terms of the cost-to-go function defined in (2.8 a). First, two equalities are derived. From the definition of $u_j^*(x_j, x_{j+1})$, in (2.9), one can write

$$f_j(x_j, u_j^*(x_j, x_{j+1})) = x_{j+1} \tag{4.1}$$

Let us assume that $u_j^*(x_j, x_{j+1})$ is differentiable. Differentiating with respect to x_{j+1} yields

$$\frac{\partial f_j(x_j, u_j)}{\partial u_j} \frac{\partial u_j^*(x_j, x_{j+1})}{\partial x_{j+1}} = 1 \tag{4.2}$$

This demonstrates that $\partial f_j(x_j, u_j)/\partial u_j$ has a right inverse. Now if (2.8 a) is treated as a parameter optimization problem of $u_{j,M-1}$, the necessary conditions for a local minimum are

$$\frac{\partial J_j^p(x_j, u_{j,M-1})}{\partial u_j} = \frac{\partial g_j(x_j, u_j)}{\partial u_j} + \frac{\partial J_{j+1}^p(x_{j+1}, u_{j+1,M-1})}{\partial x_{j+1}} \frac{\partial f_j(x_j, u_j)}{\partial u_j} = 0 \quad (4.3)$$

when evaluated at u^* . Multiplying by $\partial u_j^*(x_j, x_{j+1})/\partial x_{j+1}$ yields

$$\frac{\partial J_{j+1}^p(x_{j+1}, u_{j+1,M-1})}{\partial x_{j+1}} = - \frac{\partial g_j(x_j, u_j)}{\partial u_j} \frac{\partial u_j^*(x_j, x_{j+1})}{\partial x_{j+1}} \quad (4.4)$$

The following lemma can now be stated.

Lemma 4

When all quantities are evaluated at the optimal solution x^*, u^* , the following hold.

(a) If the quadratic incentives $I_j(x_{j+1})$ solve the problem (P - H) then

$$\begin{aligned} \frac{\partial I_j}{\partial x_{j+1}} &= p_j + Qx_{j+1} \\ &= \frac{\partial J_{j+1}^p(x_{j+1}, u_{j+1,M-1})}{\partial x_{j+1}} = - \frac{\partial g_j(x_j, u_j)}{\partial u_j} \frac{\partial u_j^*(x_j, x_{j+1})}{\partial x_{j+1}} \end{aligned} \quad (4.5)$$

(b) If $J_j^{p*}(x_{j+1})$ is C^1 then

$$\frac{\partial J_j^{p*}}{\partial x_{j+1}} = - \frac{\partial g_j(x_j, u_j)}{\partial u_j} \frac{\partial u_j^*(x_j, x_{j+1})}{\partial x_{j+1}} \quad (4.6)$$

(c) If the two conditions above hold, then

$$\frac{\partial I_j}{\partial x_{j+1}} = \frac{\partial J_j^{p*}}{\partial x_{j+1}} = \frac{\partial J_{j+1}^p(x_{j+1}, u_{j+1,M-1})}{\partial x_{j+1}} \quad (4.7)$$

Proof

The necessary conditions at the solution of (P - j) are

$$\frac{\partial J_j^i}{\partial u_j} = \frac{\partial g_j(x_j, u_j)}{\partial u_j} + \frac{\partial I_j}{\partial u_j} = 0 \quad (4.8)$$

This yields

$$\frac{\partial g_j(x_j, u_j)}{\partial u_j} + \frac{\partial I_j}{\partial x_{j+1}} \frac{\partial f_j(x_j, u_j)}{\partial u_j} = 0 \quad (4.9)$$

Similarly if J_j^{p*} is differentiable then to solve for u_j (locally) one needs to solve

$$\text{lmin}_{u_j} (g_j + J_j^{p*}) \quad (4.10)$$

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The necessary conditions become

$$\frac{\partial g_j(x_j, u_j)}{\partial u_j} + \frac{\partial J_j^*}{x_{j+1}} \frac{\partial f_j(x_j, u_j)}{\partial u_j} = 0 \quad (4.11)$$

A comparison of (4.9) and (4.11) in Lemma 4 with (4.3) and the resulting (4.4), yield the desired results. (Recall that $\partial I_j / \partial x_{j+1} = p_j + Qx_{j+1}$.)

5. Conclusion

The present work demonstrates that non-convex optimal control problems can be decomposed by using quadratic incentives. With certain technical assumptions, it is shown that the resulting high-level problem, which coordinates the low-level problems through the use of incentives p , is equivalent to the basic problem in the sense demonstrated in Theorem 1. An additional assumption about the differentiability of the solutions of subproblems yields the very interesting result that there is a one-to-one correspondence via differentiable functions between the incentive parameters p , and the conglomerate state vector x . Finally, it is demonstrated that a relationship exists between the optimal cost-to-go and incentive terms when evaluated at the optimal solution.

What remains is to demonstrate the numerical feasibility of solving the high-level problem. In Part 2 (Bromberg *et al.* 1989) of our work a gradient algorithm is used to solve the high-level problem (P – H). A higher-order method is also presented. Note that the high-level cost is defined only by solving the low-level problems sequentially. It remains to demonstrate how to decouple the solution process, by constructing a prediction method for the initial states for each subproblem. This issue is also addressed in Part 2.

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Appendix

Normally a non-convex optimal control problem is written as follows:

$$\min_{s_k} J, \quad J = \sum_{k=0}^{N-1} c_k(w_k, s_k) + c_N(w_N) \quad (A 1 a)$$

subject to the system dynamics

$$w_{k+1} = t_k(w_k, s_k), \quad k = 0, \dots, N-1 \quad (A 1 b)$$

where w_0 is given. To decompose the problem into M subproblems along the time axis, let us assume for simplicity that $N = MT$. Thus J can be rewritten as

$$J = \sum_{n=0}^{M-1} \left(\sum_{k=nT}^{(n+1)T-1} c_k(w_k, s_k) \right) + c_N(w_N) \quad (A 2)$$

Now define the following notation:

$$u_j \equiv (s_{jT}, s_{jT+1}, \dots, s_{jT+T-1}) \quad (A 3 a)$$

$$u \equiv (u_0, u_1, u_2, \dots, u_{M-1}) \in U \quad (A 3 b)$$

$$x_j \equiv w_{jT} \quad (\text{A } 3 \text{ c})$$

$$x \equiv (x_0, x_1, x_2, \dots, x_M) \in X \quad (\text{A } 3 \text{ d})$$

$$x_{j+1} = f_j(x_j, u_j) \equiv f_j(x_j, s_{jT}, s_{jT+1}, \dots, s_{jT+T-1}) \quad (\text{A } 4 \text{ a})$$

$$\equiv t_{jT+T-1}(t_{jT+T-2}(\dots t_{jT}(x_j, s_j) \dots, s_{jT+T-2}), s_{jT+T-1}) \quad (\text{A } 4 \text{ b})$$

$$g_j(x_j, u_j) \equiv \sum_{k=jT}^{(j+1)T-1} c_k(w_k, s_k), \quad 0 \leq j < M-1 \quad (\text{A } 5 \text{ a})$$

where

$$w_{k+1} = t_k(w_k, s_k), \quad jT < k < (j+1)T \quad (\text{A } 5 \text{ b})$$

and

$$q_{M-1}(x_M, u_{M-1}) \equiv \sum_{k=(M-1)T}^{MT-1} c_k(w_k, s_k) + c_N(w_N) \quad (\text{A } 6 \text{ a})$$

where

$$w_{k+1} = t_k(w_k, s_k), \quad (M-1)T \leq k < MT \quad (\text{A } 6 \text{ b})$$

Thus our problem statement can be written as problem (P) defined below, which is called the basic problem. This is to contrast the decomposed high-level and low-level problems defined in § 2.

Basic problem: (P)

$$\min_u J(u), \quad J(u) = \sum_{j=0}^{M-1} g_j(x_j, u_j) \quad (\text{A } 7)$$

such that $f_j(x_j, u_j) = x_{j+1}$.

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